Moreau sweeping process with bounded truncated retraction

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Dedicated to the memory of Jean Jacques Moreau

Abstract

The paper deals with sweeping process measure differential inclusions with data sets subject to the condition of bounded retraction along bounded truncation. Various basic properties are provided, and the existence and uniqueness of solution are established under the convexity of the data sets.

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1 Introduction

Motivated by Mechanical problems related to friction and plasticity, Jean Jacques Moreau introduced the now celebrated "Sweeping Process" in his 1971 paper "Rafle[1] par un convexe variable I" appeared in the well-known "Travaux du Séminaire d'Analyse Convexe de Montpellier", the father of "Journal of Convex Analysis". J.J. Moreau produced more than 25 papers devoted to theoretical studies and to numerical analysis of the sweeping process, and of course to its applications in "unilateral" mechanics; note that the term " unilateral [2] " (from Mechanics) was central in the title "Séminaire d'Analyse Unilatérale de Montpellier" of the 1968-69 weekly seminar of analysis, the predecessor of "Séminaire d'Analyse Convexe de Montpellier". Given an interval $I = [T_0, T]$ in $\mathbb{R}$ and for

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[1]"Sweep" in english

[2]We refer, e.g., to [29] (dated from 1917) for the mechanical aspect of this expression.

each $t \in I$ a nonempty closed subset $C(t)$ of a Hilbert space H, that is, a set-valued mapping $C : I \rightrightarrows H$ with nonempty closed values, the *sweeping process* associated with the moving set $C(t)$ is the evolution differential inclusion

$$\begin{cases} \frac{du}{dt}(t) \in -N\big(C(t); u(t)\big) \quad \text{a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(T_0) = u_0 \in C(T_0), \end{cases} \tag{1.1}$$

where $N\big(C(t); \cdot\big)$ denotes a general normal cone to the set $C(t)$. In the first 1971 paper [51], J.J. Moreau investigated the existence of an absolutely continuous solution $u(\cdot)$, that is, an absolutely continuous mapping $u : I \to H$ satisfying the initial condition and whose derivative (which exists Lebesgue almost everywhere, thanks to the Radon-Nikodym property of Hilbert spaces) fulfills the inclusion in (1.1) with the normal cone for Lebesgue almost all $t \in I$. Obviously, if $C(t)$ is reduced to a singleton for every $t \in I$, the existence of such an absolutely continuous solution requires $C(\cdot)$ to be absolutely continuous. So, in [51] the sweeping process is studied under the convexity of $C(t)$ and under the assumption that the set-valued mapping $C(\cdot)$ is absolulety continuous with respect to the Hausdorff-Pompeiu distance, which is equivalent to the existence of an absolutely continuous function $v : I \to \mathbb{R}$ such that

$$\text{haus}\big(C(s), C(t)\big) \le |v(s) - v(t)| \quad \text{for all } s, t \in I.$$

Denoting by f_t the indicator function of the convex set $C(t)$ (i.e. $f_t(x) = 0$ if $x \in C(t)$ and $f_t(x) = +\infty$ if $x \in H \setminus C(t)$), the *Moreau λ-envelope $e_\lambda f_t$* (with $\lambda > 0$) is known to be differentiable on H with its derivative Lipschitz continuous on every bounded subset of H (see [49, 13, 70]), and here

$$e_\lambda f_t(x) = \inf_{y \in H} \big(f_t(y) + \frac{1}{2\lambda}\|x - y\|^2\big) = \frac{1}{2\lambda} d^2_{C(t)}(x).$$

For each fixed $t \in I$, the Moreau λ-envelope $e_\lambda f_t$ is also known to converge as $\lambda \downarrow 0$ to f_t in the suitable sense of epiconvergence (or Γ-convergence) while the graph of $\nabla e_\lambda f_t$ converges to the graph of $N(C(t); \cdot)$ in the Painlevé-Kuratowski sense. The approach in [51] (and also in the survey paper [53]) consists in a first step in approximating the differential inclusion (1.1) with the usual differential equation

$$\begin{cases} \frac{du_\lambda}{dt}(t) = -\frac{1}{2\lambda}\nabla\big(d^2_{C(t)}\big)(u_\lambda(t)) \\ u_\lambda(T_0) = u_0, \end{cases}$$

and after a suitable long analysis of the properties of the solution $u_\lambda(\cdot)$ of the latter differential equation, in demonstrating in a second step that $(u_\lambda(\cdot))_\lambda$ converges uniformly on I as $\lambda \downarrow 0$ to an absolutely continuous mapping $u : I \to H$ which is shown to be a solution of (1.1).

As observed by J.J. Moreau, if the set-valued mapping $C(\cdot)$ is non-decreasing, that is, $C(s) \subset C(t)$ for all $s < t$ in I, it is evident (with $u_0 \in C(T_0)$) that the constant mapping $t \mapsto u_0$ is a solution of (1.1), so in such a case no Hausdorff-Pompeiu absolute continuity of $C(\cdot)$ is needed. This observation led J.J. Moreau to merely assume in the 1972 paper [52] and in [56] that the set-valued mapping $C(\cdot)$ is of bounded retraction. Under the latter assumption and the convexity of $C(t)$, it is established in [52, 56] the existence and uniqueness of a BV solution for (1.1). Set-valued mapping of bounded retraction is studied in great detail in Moreau [54]. The retraction of the set-valued mapping $C(\cdot)$ on the interval I is defined as

$$\operatorname{ret}(C; I) := \sup\{\sum_{i=0}^{n-1} \sup_{x \in C(t_i)} d(x, C(t_{i+1}))\},$$

where the first supremum is taken over all finite sequences $t_0 < t_1 < \cdots < t_n$ in I. When the retraction $\operatorname{ret}(C; I)$ is finite, one says that the set-valued mapping $C(\cdot)$ is *of bounded retraction* (or *of finite retraction*) on I. To prove the existence of a solution with bounded variation of (1.1), under the bounded retraction assumption of the convex set $C(t)$, J.J. Moreau introduced in [52, 56] the efficient "Catching-up Algorithm" and showed its convergence, with respect to the uniform convergence, towards a mapping $u(\cdot)$ of bounded variation which is a solution of (1.1). In view of that he defined in such a context the concept of solution through the differential measure associated with the mapping of bounded variation $u(\cdot)$ (see the next section). This solution concept has practical applications in Mechanical problems and it possesses a good behavior concerning numerical aspects (see [57]).

Many other deep papers are also born from other members of the famous 80's team of the "Laboratoire d'Analyse Convexe de Montpellier". A stochastic version of (1.1) was developed by C. Castaing [15, 16] with closed convex set $C(t, \omega)$ and a probabilty space $(\Omega, \mathcal{A}, P)$. Additional existence results for Convex Sweeping Processes have been provided by M.D.P. Monteiro Marques (see [47] for references), in particular the existence of BV solution is established when $C(\cdot)$ is Hausdorff-Pompeiu continuous and all the closed convex sets $C(t)$ contain a fixed ball. Existence of solutions is also proved in Monteiro Marques [46] when external forces are applied (through a set-valued mapping) to the system whose sweeping process is a mathematical modeling; more precisely, with an upper semicontinuous set-valued mapping $F : I \times H \rightrightarrows H$ with nonempty compact convex values the existence of absolutely continuous solutions is demonstrated in [46], under the convexity of $C(t)$ and other appropriate assumptions, for the evolution differential inclusion

$$\begin{cases} \frac{du}{dt}(t) \in -N\big(C(t); u(t)\big) - F(t, u(t)) \quad \text{a.e. } t \in I, \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(T_0) = u_0 \in C(T_0). \end{cases} \tag{1.2}$$

Periodic solutions of the latter differential inclusion are investigated by C. Castaing and M.D.P. Monteiro Marques [19]. The study of Nonconvex Sweeping Processes started with M. Valadier [75, 76, 77]. Those papers of Valadier considered the existence of absolutely continuous solutions when $C(t) = \mathbb{R}^d \setminus \operatorname{int} K(t)$, where $K(t)$ is a convex set in $\mathbb{R}^d$ with nonempty interior (in fact with a more general class of subsets $C(t)$ in $\mathbb{R}^d$). Then, C. Castaing, T.X. Duc Ha and M.

Valadier [18] and C. Castaing and M.D.P. Monteiro Marques [20] extensively studied (1.2) when $C(t)$ is the complement of the interior of a variable convex set $K(t)$.

Existence and properties of absolutely continuous solutions of (1.2) with a fixed convex (or tangentially regular) set $C(t) = S \subset \mathbb{R}^d$ have been also studied in [28, 37] in view of the analysis of resource allocation mechanisms in Economics; see also [33] for another work with application in Economics. Besides various Mechanical Problems (see, e.g., [53, 41, 47, 61]) and Resource Allocation in Economics (see, e.g., [7, 28, 37]) Sweeping Processes also arise in Thermo-Plasticity and Phase Transition Problems (see, e.g., [38, 39, 40, 66, 79]), in Nonregular Electrical Circuits (see, e.g., [1, 2, 3, 14]), in Crowd Motion Modeling (see, e.g., [43, 44, 78]), in Variational Inequalities (see, e.g., [60, 65, 3, 45]), in (Differential) Complementarity Problems (see, e.g., [14, 3, 4]).

When $C(t)$ is any closed subset of $\mathbb{R}^d$ moving with an absolutely continuous variation, the existence of absolutely continuous solutions has been established (with the Clarke normal cone to $C(t)$) in [8, 24, 72] for (1.1), and in [72] for (1.2). The case of a prox-regular set $C(t)$ of a Hilbert space is actually very active. For studies with a prox-regular set $C(t)$ in a Hilbert space moving in an absolutely continuous way we refer, e.g., to [24, 72, 12, 31, 26, 35, 36, 4, 14, 9, 71, 73, 62] and the references therein, and for the case when the prox-regular set $C(t)$ has a bounded variation we refer to [32, 14, 4]. For stochastic versions of the sweeping process involving a Brownian motion we cite F. Bernicot and J. Venel [10] and C. Castaing, M.D.P. Monteiro Marques and P. Raynaud de Fitte [21] in this volume; see also H. Frankowska [34].

Besides the case $\mathrm{ret}(C; I) < +\infty$ which is natural in mechanics, coming back to (1.1) with a closed convex set $C(t)$ of a Hilbert space, it has been recenly observed by G. Colombo, R. Henrion, N.D. Hoan and B.S. Mordukhovich [25] and by A.A. Tolstonogov [74] that there are concrete and practical situations where the retraction of the convex-valued mapping $C(\cdot)$ on the interval I *is not bounded*. Nevertheless, under relaxed retraction absolute continuity properties, those authors proved in [74, 25] the existence of absolutely continuous solution for (1.1). The aim of the present paper is to show how the Moreau Catching-up Algorithm can be still used to establish the existence of BV solution of the measure differential inclusion on I

$$\begin{cases} du \in -N\big(C(t); u(t)\big), \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(T_0) = u_0 \in C(T_0), \end{cases} \tag{1.3}$$

where $C : I \rightrightarrows H$ is a set-valued mapping with nonempty closed convex subsets of a Hilbert space H such that some *truncated retraction* of $C(\cdot)$ is bounded. The truncated retraction of $C(\cdot)$ and its boundedness are defined in Section 2 and related preliminaries are given. In Section 3 various basic properties of solutions are provided even in the nonconvex setting. In this same section, the uniqueness of solution is established and the situation when the truncated retraction is absolutely continuous is examined. The existence of solution is achieved in Section 4.

2 Normal cone and retraction of set-valued mapping along truncation

Throughout H will be a (real) Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and $B(x, r)$ (resp. $B[x, r]$) the *open ball* (resp. *closed ball*) centered at $x \in H$ with radius $r > 0$. It will be also convenient to denote by $\mathbb{B}$ the closed unit ball around zero, that is, $\mathbb{B} := B[0, 1]$. The distance from a point $x \in H$ to a nonempty subset S of H is denoted by $d(x, S)$ or $d_S(x)$, that is, $d(x, S) = \inf_{y \in S} \|x - y\|$; by convention $d(x, S) = +\infty$ when $S = \emptyset$. By $\mathbb{N}$ we will mean as usual the set of integers starting from 1.

2.1 Normal cone and subdifferential in Convex Analysis

Assuming that the set $S \subset H$ is convex, its *normal cone* at $x \in S$ is defined by

$$N(S; x) := \{\zeta \in H : \langle \zeta, x' - x \rangle \leq 0, \ \forall x' \in S\}.$$

So, in particular when S is closed and convex

$$y - \operatorname{proj}(y, S) \in N\big(S; \operatorname{proj}(y, S)\big) \quad \text{for all } y \in H, \tag{2.1}$$

where $\operatorname{proj}(y, S)$ denotes the nearest point of y into S; consequently, one has the equivalence

$$y - x \in N(S; x) \iff x = \operatorname{proj}(y, S) \tag{2.2}$$

(sometimes, it will also be convenient to write $\operatorname{proj}_S(y)$ in place of $\operatorname{proj}(y, S)$). One also defines $N(S; x) = \emptyset$ whenever $x \notin S$. The normal cone is linked with the subdifferential concept. Given a convex function $f : H \to \mathbb{R} \cup \{+\infty\}$ we recall (see [58, 68, 69, 22]) that its *subdifferential* at a point $x \in H$ where f is finite is given by

$$\partial f(x) := \{\zeta \in H : \langle \zeta, x' - x \rangle \leq f(x') - f(x), \ \forall x' \in H\},$$

and $\partial f(x) = \emptyset$ if $f(x)$ is not finite. Defining the indicator function of the set S by

$$\psi_S(x) = 0 \quad \text{if } x \in S \quad \text{and} \quad \psi_S(x) = +\infty \quad \text{if } x \in H \setminus S,$$

it is clear that $N(S; x) = \partial \psi_S(x)$ in the above sense when S is convex.

For the convex function f finite at x, the directional derivative of f in any direction $h \in H$

$$f'(x; h) := \lim_{t \downarrow 0} t^{-1}\big(f(x + th) - f(x)\big) = \inf_{t > 0} t^{-1}\big(f(x + th) - f(x)\big) \tag{2.3}$$

always exists in $\mathbb{R} \cup \{-\infty, +\infty\}$ and the function $f'(x; \cdot)$ is convex, and it is also known (and easy to see) that

$$\partial f(x) = \{\zeta \in H : \langle \zeta, h \rangle \leq f'(x; h), \ \forall h \in H\}.$$

If the set S is convex, the distance function d_S is convex, and the normal cone of S at $x \in S$ is linked with the function d_S through the following equalities that will be used in the development of the paper:

$$N(S; x) = \mathbb{R}_+ \partial d_S(x) \quad \text{and} \quad \partial d_S(x) = N(S; x) \cap \mathbb{B}. \tag{2.4}$$

2.2 Proximal normal cone and proximal subdifferential

Given a nonempty subset S of the Hilbert space H, let us denote by $\operatorname{Proj}(y, S)$ the set of nearest points from $y \in H$ in S, that is,

$$\operatorname{Proj}(y, S) := \{z \in S : \|x - y\| = d_S(y)\}.$$

A vector $\zeta \in H$ is a *proximal normal vector* to S at $x \in S$ (see, e.g., [23]) provided that there is some real $\sigma > 0$ such that $x \in \operatorname{Proj}(x + \sigma\zeta)$, which is equivalent to

$$x = \operatorname{proj}(x + t\zeta) \quad \text{for all } 0 \le t < \sigma.$$

The set of such vectors ζ is denoted by $N^P(S; x)$ and called the *proximal normal cone* of S at $x \in S$. It is usual and convenient to set $N^P(S; x) = \emptyset$ whenever $x \notin S$. If S is closed and convex, $N^P(S; x)$ coincides with the normal cone in the previous subsection for convex sets.

It is worth noting that, whenever $\operatorname{Proj}(y, S) \neq \emptyset$, one has

$$y - z \in N^P(S; z) \quad \text{for all } z \in \operatorname{Proj}(S, y),$$

extending the inclusion in (2.1) to nonconvex sets.

For a function $f : H \to \mathbb{R} \cup \{+\infty\}$ which is finite at $x \in H$, a vector $\zeta \in H$ is a *proximal subgradient* of f at x if there exist a real $\sigma > 0$ and a neighborhood U of x such that

$$\langle \zeta, x' - x\rangle \le f(x') - f(x) + \frac{\sigma}{2}\|x' - x\|^2 \quad \text{for all } x' \in U.$$

The set of such vectors is denoted by $\partial_P f(x)$ and called the *proximal subdifferential* of f at x. As above one also puts $\partial_P f(x) = \emptyset$ when f is not finite at the point x. Assuming that S is closed and using the indicator function ψ_S, the known equality $N^P(S; x) = \partial_P \psi_S(x)$ provides the following practical analytical characterization of the proximal normal cone: $\zeta \in N^P(S; x)$ if and only there are a real $\sigma > 0$ and a neighborhhod U of $x \in S$ such that

$$\langle \zeta, x' - x\rangle \le \frac{1}{2}\sigma\|x' - x\|^2 \quad \text{for all } x' \in U \cap S.$$

Both equalities in (2.4) still hold (see, e.g., [11]), with the proximal subfifferential and normal cone, for any closed subset S of H and $x \in S$,

$$N^P(S; x) = \mathbb{R}_+ \partial_P d_S(x) \quad \text{and} \quad \partial_P d_S(x) = N^P(S; x) \cap \mathbb{B}. \tag{2.5}$$

2.3 Retraction along truncation of set-valued mapping

Given an extended real $\rho \in]0, +\infty]$ and two subsets S and S' of H with $S' \neq \emptyset$, the *ρ-excess of S over S'* (or the *excess of the ρ-truncation of S over S'*) is defined by

$$\operatorname{exc}_\rho(S, S') := \sup_{x \in S \cap \rho\mathbb{B}} d(x, S'),$$

where as usual $\sup_{x\in Q} d(x,S') = 0$ if $Q = \emptyset$. In the equality for the truncated excess, with $\rho = +\infty$ it is taken by convention $\rho\mathbb{B} = H$, so in this case the ρ-excess coincides with the usual excess of S over S' denoted by $\mathrm{exc}(S,S')$; this equivalently means that

$$\mathrm{exc}_\infty(S,S') = \sup_{x\in S} d(x,S') =: \mathrm{exc}(S,S').$$

Symmetrizing the ρ-excess yields the *Hausdorff ρ-semidistance*

$$\mathrm{Haus}_\rho(S,S') := \max\{\mathrm{exc}_\rho(S,S'), \mathrm{exc}_\rho(S',S)\};$$

when $\rho = +\infty$ the latter formula is reduced to the usual Hausdorff-Pompeiu distance and as above we write $\mathrm{Haus}(S,S')$.

It is worth noting that, for every $x' \in H$

$$d(x',S') \le d(x',x) + \mathrm{exc}_\rho(S,S') \quad \text{for all } x \in S \cap \rho\mathbb{B},$$

or equivalently

$$d(x',S') \le d(x',S\cap\rho\mathbb{B}) + \mathrm{exc}_\rho(S,S'). \tag{2.6}$$

It is also known and not difficult to show (see, e.g., [6, 70]) that

$$\mathrm{exc}_\rho(S,S') = \inf\{r > 0 :\ S\cap\rho\mathbb{B} \subset S' + r\mathbb{B}\},$$

and that

$$\mathrm{exc}_\rho(S,S') \le \sup_{x\in\rho\mathbb{B}} \Big(\max\{d(x,S') - d(x,S), 0\} \Big)$$

and with $\rho' \ge 2\rho + d(0,S_1)$

$$\sup_{x\in\rho\mathbb{B}} \Big(\max\{d(x,S') - d(x,S), 0\} \Big) \le \mathrm{exc}_{\rho'}(S,S').$$

In the paper, given two reals $T_0 < T$ and $I := [T_0,T]$ we are concerned on I with the *measure differential inclusion*

$$\begin{cases} du \in -N(C(t);u(t)) \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0 \in C(0), \end{cases} \tag{2.7}$$

where the way of moving of the set $C(t) \subset H$ is translated by a given positive Radon measure. In fact, we will often have to consider an extended real $\rho \in [\|u_0\|, +\infty]$ and a positive Radon measure μ on $I = [T_0,T]$ such that

$$\mathrm{exc}_\rho\big(C(s),C(t)\big) \le \mu(]s,t]) \quad \text{for all } s,t \in I \text{ with } s \le t. \tag{2.8}$$

The existence of such a measure μ is related to the concept of retraction of $C(\cdot)$ along ρ-truncation. Given a nonempty interval $\mathcal{I}$ of $\mathbb{R}$ and a set-valued mapping $C : \mathcal{I} \rightrightarrows H$, the *retraction along ρ- truncation of $C(\cdot)$ on $\mathcal{I}$* or the

ρ-retraction of $C(\cdot)$ on $\mathcal{I}$, denoted by $\mathrm{ret}_\rho(C;\mathcal{I})$, is defined as the supremum of the sums

$$\sum_{i=0}^{n-1} \mathrm{exc}_\rho\big(C(t_i), C(t_{i+1})\big)$$

over all finite sequences $t_0 < \cdots < t_n$ of $\mathcal{I}$. When $\mathrm{ret}_\rho(C;\mathcal{I})$ is finite (resp. $\mathrm{ret}_\rho(C;\mathcal{J})$ is finite for every compact interval $\mathcal{J}$ included in $\mathcal{I}$), we say that $C(\cdot)$ has on $\mathcal{I}$ a *bounded ρ-retraction* or *a bounded retraction along ρ-truncation* (resp. *a locally bounded ρ-retraction* or *a locally bounded retraction along ρ-truncation*). If $\rho = +\infty$ the above concepts amount to saying, in the sense of Moreau [54], that $C(\cdot)$ has a bounded retraction (resp. a locally bounded retraction) on $\mathcal{I}$. Replacing the ρ-excess by the Hausdorff ρ-semidistance defines on the one hand the *ρ-variation of $C(\cdot)$ on $\mathcal{I}$* or the *variation of $C(\cdot)$ on $\mathcal{I}$ along ρ-truncation* denoted by $\mathrm{var}_\rho(C;\mathcal{I})$, and on the other hand the concept of set-valued mapping with *bounded ρ-variation* or with *bounded variation along ρ-truncation* (resp. with *locally bounded ρ-variation* or with *locally bounded variation along ρ-truncation*); the case when $\rho = +\infty$ translates the standard concept of set-valued mapping with bounded variation (resp. locally bounded variation).

Clearly, the boundedness (resp. local boundedness) of the retraction of $C(\cdot)$ entails the boundedness (resp. local boundedness) of the retraction along ρ-truncation, and the former property is implied by the bounded (resp. locally bounded) variation property.

When $\mathcal{J} = [s,t]$ with $s < t$, instead of using $[s,t]$ in the above notations we will write $\mathrm{ret}_\rho(C;s,t)$ and $\mathrm{ret}(C;s,t)$, and also $\mathrm{var}_\rho(C;s,t)$ and $\mathrm{var}(C;s,t)$. So, coming back to (2.8) we see that the existence of such a positive Radon measure μ on the interval $I = [T_0, T]$ (combined with the property that the measure of any compact set under a Radon measure is finite) guarantees that the set-valued mapping $C(\cdot)$ has bounded retraction on I along ρ-truncation and that the function $t \mapsto \mathrm{ret}_\rho(C;T_0,t)$ is right continuous on I since given $\bar{t} \in [T_0, T[$ it holds for all $t \in]\bar{t}, T[$

$$0 \le \mathrm{ret}_\rho(C;T_0,t) - \mathrm{ret}_\rho(C;T_0,\bar{t}) \le \mu(]\bar{t},t]);$$

furthermore,

$$\mathrm{ret}_\rho(C;T_0,T) \le \mu(]T_0,T]).$$

Conversely, if we assume that the set-valued mapping $C(\cdot)$ has a bounded retraction on I along ρ-truncation and that the function $t \mapsto \mathrm{ret}_\rho(C;T_0,t)$ is right-continuous, then the latter function is right-continuous and with bounded variation on I (in fact, non-decreasing on I). So, if we denote by $\mu_{C,\rho}$ the differential Radon measure associated with it (see (2.10) below) we have

$$\mathrm{ret}_\rho(C;T_0,t) - \mathrm{ret}_\rho(C;T_0,s) = \mu_{C,\rho}(]s,t]) \quad \text{for all } s \le t \text{ in } I,$$

which yields

$$\mathrm{exc}_\rho\big(C(s),C(t)\big) \le \mu_{C,\rho}(]s,t]) \quad \text{for all } s \le t \text{ in } I,$$

otherwise stated, condition (2.8) is satisfied with the positive Radon measure $\mu_{C,\rho}$ in place of μ.

2.4 Concept of solution

In addition to the above notions, some other ones are necessary before defining the concept of solution of the measure differential inclusion (2.7). Troughout the rest of the paper, any positive measure μ (resp. vector measure m) on the compact interval $I := [T_0, T]$ of $\mathbb{R}$ will be a Radon measure; in particular $\mu([T_0, T])$ is finite. We will use the notation $r \downarrow 0$ to mean $r \to 0$ with $r > 0$. The function $\mathbf{1}_A$ will denote the characteristic function of a subset $A \subset I$, that is, $\mathbf{1}_A(t) = 1$ if $t \in A$ and $\mathbf{1}_A(t) = 0$ otherwise.

Given a positive Radon measure ν on the interval I and a real $p \in [1, +\infty[$, we will denote by $L^p_\nu(I, H)$ the space of all (classes of) ν-measurable mappings $u : I \to H$ such that $t \mapsto \|u(t)\|^p$ is ν-integrable on I.

We start by recalling some results from real and vector measures. For two positive Radon measures ν and $\widehat{\nu}$ on I and for $I(t, r) := I \cap [t - r, t + r]$ with $r > 0$, it is known (see, e.g., [42, Theorem 2.12]) that the limit

$$\frac{d\widehat{\nu}}{d\nu}(t) := \lim_{r \downarrow 0} \frac{\widehat{\nu}(I(t, r))}{\nu(I(t, r))}$$

(with the convention $\frac{0}{0} = 0$) exists in $\mathbb{R}$ for ν-almost every $t \in I$ and it defines a Borel function of t, called the *derivative* of $\widehat{\nu}$ with respect to ν. Furthermore, the measure $\widehat{\nu}$ is absolutely continuous with respect to ν if and only if $\frac{d\widehat{\nu}}{d\nu}(\cdot)$ is a density of $\hat{\nu}$ relative to ν, or otherwise stated, if and only if the equality $\widehat{\nu} = \frac{d\widehat{\nu}}{d\nu}(\cdot)\nu$ holds true. Under such an absolute continuity assumption, a mapping $u(\cdot) : I \to H$ is $\widehat{\nu}$-integrable on I if and only if the mapping $t \mapsto u(t)\frac{d\widehat{\nu}}{d\nu}(t)$ is ν-integrable on I; furthermore, in that case,

$$\int_I u(t)\, d\widehat{\nu}(t) = \int_I u(t) \frac{d\widehat{\nu}}{d\nu}(t)\, d\nu(t). \tag{2.9}$$

When ν and $\widehat{\nu}$ are each one absolutely continuous with respect to the other, we will say that they are *absolutely continuously equivalent*.

Given a vector measure m on I with values in the Hilbert space H, its *variation measure* $|m|$ is defined for any Borel set $A \subset I$ by

$$|m|(A) := \sup \sum_{n \in \mathbb{N}} \|m(B_n)\|,$$

where the supremum is taken over all sequences $(B_n)_{n \in \mathbb{N}}$ of Borel mutually disjoint subsets of I such that $A = \bigcup_{n \in \mathbb{N}} B_n$ (see, e.g., [30]). When the positive measure $|m|$ is absolutely continuous with respect to the positive measure ν, we will say that the vector measure m is *absolutely continuous with respect to ν*. In such a case, since the Hilbert space H has the Radon-Nikodým property, the vector measure m has a density $\zeta : I \to H$ relative to ν, that is, $m = \zeta(\cdot)\nu$, or equivalently $\zeta(\cdot)$ is ν-integrable on I and

$$m(A) = \int_A \zeta(t)\, d\nu(t) \quad \text{for all Borel sets } A \subset I.$$

Now consider a mapping $u(\cdot) : I \to H$ of *bounded variation* and denote by du the differential vector measure associated with u (see [30] and [55]); if in addition, $u(\cdot)$ is right continuous, then

$$u(t) = u(s) + \int_{]s,t]} du \quad \text{for all } s, t \in I \text{ with } s \le t. \tag{2.10}$$

Conversely, if there exists some mapping $\widehat{u}(\cdot) \in L^1_\nu(I, H)$ such that $u(t) = u(T_0) + \int_{]T_0,t]} \widehat{u}\, d\nu$ for all $t \in I$, then $u(\cdot)$ is of bounded variation and right continuous; for the associated differential vector measure du it is known that its variation measure $|du|$ satisfies

$$|du|(]s, t]) = \int_{]s,t]} \|\widehat{u}(\tau)\|\, d\nu(\tau) \quad \text{for all } s, t \in I \text{ with } s \le t,$$

and du is absolutely continuous with respect to ν and admits $\widehat{u}(\cdot)$ as a density relative to ν, that is,

$$du = \widehat{u}(\cdot)\, d\nu.$$

Then, putting $I^-(t, r) := [t - r, t] \cap I$ and $I^+(t, r) := [t, t + r] \cap I$, according to Moreau and Valadier [59], for ν-almost every $t \in I$, the following limits exist in H and

$$\widehat{u}(t) = \frac{du}{d\nu}(t) := \lim_{r \downarrow 0} \frac{du(I(t,r))}{d\nu(I(t,r))} = \lim_{r \downarrow 0} \frac{du(I^-(t,r))}{d\nu(I^-(t,r))} = \lim_{r \downarrow 0} \frac{du(I^+(t,r))}{d\nu(I^+(t,r))}, \tag{2.11}$$

and from this it is easy to see that one also has, for ν-almost every $t \in I$

$$\widehat{u}(t) = \frac{du}{d\nu}(t) = \lim_{r \downarrow 0} \frac{du(]t - r, t] \cap I)}{d\nu(]t - r, t] \cap I)}, \tag{2.12}$$

where, for convenience as usual $d\nu(A) := \nu(A)$ for every ν-measurable subset $A \subset I$. In particular, it ensues that

$$\frac{du}{d\nu}(t) = \frac{du(\{t\})}{d\nu(\{t\})} \quad \text{and} \quad \frac{d\lambda}{d\nu}(t) = 0, \ \text{whenever } \nu(\{t\}) > 0. \tag{2.13}$$

Above and in the rest of the paper λ denotes the *Lebesgue measure.*

As usual, for the variation measure $|du|$ of the vector differential measure du associated with the bounded variation mapping $u : I \to H$, it will be convenient to denote by

$$\frac{du}{|du|} \quad \text{in place of} \quad \frac{du}{d(|du|)}$$

the derivative of du with respect to $|du|$.

Restricting our study (as said above)to the framework where (2.8) is valid, our definition of solution of (2.7) is as follows:

Definition 2.1 *Let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values and let $N(C(t);\cdot)$ a general notion of normal cone (see, e.g. [23, 48, 63, 70]). We say that a mapping $u : [T_0, T] \to H$ is a solution of the measure differential inclusion* (2.7) *provided there are an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ on $I := [T_0, T]$ satisfying* (2.8) *and such that the following properties (i) and (ii) are fulfilled:*
(i) the mapping $u(\cdot)$ is of bounded variation on $I = [T_0, T]$, right continuous, and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;
(ii) there exists a positive Radon measure ν absolutely continuously equivalent to μ and with respect to which the differential measure du of $u(\cdot)$ is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1_\nu(I, H)$-density and

$$\frac{du}{d\nu}(t) \in -N(C(t); u(t)) \quad \text{for } \nu - a.e.\ t \in I. \tag{2.14}$$

Some situations can lead to work with an additional (non identically λ-null) set-valued mapping $F : I \times H \rightrightarrows H$ such that, for each λ-integrable mapping $z : I \to H$, the set-valued mapping $t \mapsto F(t, z(t))$ admits a λ-integrable selection. So, instead of the measure differential inclusion (2.7) one is interested, again under the ρ-retraction assumption (2.8), to the following one

$$\begin{cases} du \in -N(C(t); u(t)) - F(t, u(t) \\ u(t) \in C(t) \quad \text{for all } t \in I, \\ u(0) = u_0 \in C(0). \end{cases} \tag{2.15}$$

Extending Definition 2.1, a solution of the latter differential inclusion is defined as follows:

Definition 2.2 *With $C(\cdot)$ and $N\big(C(t);\cdot)$ as in Definition 2.1, we say that a mapping $u : I \to H$ is a solution of the measure differential inclusion* (2.15) *if there are an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ on $I := [T_0, T]$ satisfying* (2.8) *and such that the following properties (i) and (ii) are fulfilled:*
(i) the mapping $u(\cdot)$ is of bounded variation on $I = [T_0, T]$, right continuous, and satisfies $u(T_0) = u_0$ and $u(t) \in C(t)$ for all $t \in I$;
(ii) there exist a λ-integrable selection $y_u(\cdot)$ of $t \mapsto F(t, u(t))$ and a positive Radon measure ν absolutely continuously equivalent to $\mu + \lambda$ and with respect to which the differential measure du of $u(\cdot)$ is absolutely continuous with $\frac{du}{d\nu}(\cdot)$ as an $L^1_\nu(I, H)$-density and

$$\frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \in -N(C(t); u(t)) \quad \text{for } \nu - a.e.\ t \in I.$$

As in [32], fixing a pair (ρ, μ) as above we can see that both above concepts of solutions associated to the pair (ρ, μ) do not depend on the Radon measure ν on I absolutely continuously equivalent to $\mu + \lambda$ (resp. μ), in the sense that, for any other Radon measure ν_0 on I absolutely continuously equivalent to $\mu + \lambda$ (resp. μ), the differential measure du is absolutely continuous with respect to

ν_0 and the same inclusion in Definition 2.2 (resp. Definition 2.1) is fulfilled with ν_0 in place of ν. In fact we will show that, given any positive Radon measure ν_0 on I *with respect to which* $\mu+\lambda$ *(resp.* μ*) is absolutely continuous*, the inclusion in Definition 2.2 (resp. Definition 2.1) holds true with ν_0 in place of ν. Indeed, let be given a positive Radon measure ν on I absolutely continuously equivalent to $\mu+\lambda$ and let $y_u(\cdot)$ be the associated λ-integrable selection of $t \mapsto F(t, u(t))$ such that the properties in (ii) of Definition 2.2 are fulfilled (resp. absolutely continuously equivalent to μ and such that the properties in (ii) of Definition 2.1 are fulfilled). Take any other positive Radon measure ν_0 on I with respect to which the measure ν is absolutely continuous. By property (ii), the differential measure du and the real measures λ are absolutely continuous with respect to ν. Therefore, (by the Radon Nikodým property of H) there exists a Borel subset B of I with $\nu_0(B) = 0$ such that, for all $t \in I \setminus B$

$$\frac{du}{d\nu_0}(t) = \frac{du}{d\nu}(t)\frac{d\nu}{d\nu_0}(t) \quad \text{and} \quad \frac{d\lambda}{d\nu_0}(t) = \frac{d\lambda}{d\nu}(t)\frac{d\nu}{d\nu_0}(t)$$

$$(\text{resp.} \quad \frac{du}{d\nu_0}(t) = \frac{du}{d\nu}(t)\frac{d\nu}{d\nu_0}(t) \ ).$$

Choose also a Borel subset B_1 of I with $\nu(B_1) = 0$ such that the inclusion in Definition 2.2 (resp Definition 2.1) is satisfied for all $t \in I \setminus B_1$. According to the absolute continuity of ν with respect to ν_0, we can write

$$0 = \nu(B_1) = \int_{B_1} \frac{d\nu}{d\nu_0}(t)\, d\nu_0(t).$$

Consequently, there is a Borel subset $B_2 \subset B_1$ of I with $\nu_0(B_2) = 0$ such that

$$\frac{d\nu}{d\nu_0}(t) = 0 \quad \text{for all } t \in B_1 \setminus B_2,$$

hence, for all $t \in (I \setminus B) \cap (B_1 \setminus B_2)$

$$\frac{du}{d\nu_0}(t) = 0 \quad \text{and} \quad \frac{d\lambda}{d\nu_0}(t) = 0 \quad (\text{resp } \frac{du}{d\nu_0}(t) = 0).$$

Putting $B_3 := B \cup B_2$ and using the conical property of $N(\cdot;\cdot)$ it results that $\nu_0(B_3) = 0$ and for all $t \in I \setminus B_3$

$$\frac{du}{d\nu_0}(t) + y_u(t)\frac{d\lambda}{d\nu_0}(t) \in -N(C(t); u(t)) \tag{2.16}$$

$$(\text{resp.} \quad \frac{du}{d\nu_0}(t) \in -N(C(t); u(t)) \ ), \tag{2.17}$$

as announced.

3 Basic properties of solutions

Let us start with the situation where the measure μ is absolutely continuous with respect to the Lebesgue measure λ. Consider first the case where no external force is present.

Proposition 3.1 *Let $N\big(\cdot;\cdot\big)$ be a general normal cone and, with $I := [T_0, T]$, let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values. Let $u : I \to H$ be a solution of* (2.7) *in the sense of Definition 2.1, associated with an extended real $\rho\|u_0\|$ and a positive Radon measure μ satisfying* (2.8)*. If μ is absolutely continuous with respect to the Lebesgue measure λ on I, then $u(\cdot)$ is a usual absolutely continuous solution in the sense that $u(\cdot)$ is absolutely continuous on I, $u(T_0) = u_0$, $u(t) \in C(t)$ for all $t \in I$, and*

$$\frac{du}{dt}(t) \in -N\big(C(t); u(t)\big) \quad \textit{for } \lambda - a.e.\ t \in I,$$

where $\dfrac{du}{dt}(t)$ denotes the standard derivative at point t where it exists.

Proof. Let $u(\cdot)$ be a solution in the sense of Definition 2.1 associated with a pair (ρ, μ) satisfying (2.8). Assume that the measure μ is absolutely continuous with respect to λ. Since the differential measure du is absolutely continuous with respect to μ, it is also absolutely continuous with respect to λ. By the Radon-Nikodym property of the Hilbert space H the standard derivative $\dfrac{du}{dt}(\cdot)$ exists λ-almost everywhere. We also know that $\dfrac{du}{dt}(\cdot)$ is a denstity in $L^1_\lambda(I, H)$ of du relative to λ, and that there exists some Borel set $B_1 \subset I$ with $\lambda(B_1) = 0$ and such that, for all $t \in I \setminus B_1$

$$\frac{du}{dt}(t) = \frac{du}{d\lambda}(t).$$

Further, by (2.17) there is also a Borel subset B_2 of I with $\lambda(B_2) = 0$ such that

$$\frac{du}{d\lambda}(t) \in -N\big(C(t); u(t)\big) \quad \text{for all } t \in I \setminus B_2.$$

It ensues that $\lambda(B_1 \cup B_2) = 0$ and

$$\frac{du}{dt}(t) \in -N\big(C(t); u(t)\big) \quad \text{for all } t \in I \setminus (B_1 \cup B_2),$$

So, $u(\cdot)$ is a solution in the usual sense. ■

The similar result in presence of external forces is also valid.

Proposition 3.2 *Let $N\big(\cdot;\cdot\big)$ be a general normal cone and, with $I := [T_0, T]$, let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values. Let* *also $F : I \times H \rightrightarrows H$ be a set-valued mapping as above. Let $u : I \to H$ be a solution of* (2.15) *in the sense of Definition 2.1, associated with an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ satisfying* (2.8). *If μ is absolutely continuous with respect to the Lebesgue measure λ on I, then $u(\cdot)$ is a usual absolutely continuous solution in the sense that $u(\cdot)$ is absolutely continuous on I with $u(T_0) = u_0$, $u(t) \in C(t)$ for all $t \in C(t)$, and there exists a λ-integrable selection $y_u(\cdot)$ of $t \mapsto F(t, u(t))$ such that*

$$\frac{du}{dt}(t) + y_u(t) \in -N\big(C(t); u(t)\big) \quad \textit{for } \lambda - \textit{a.e. } t \in I.$$

Proof. Let $u(\cdot)$ be a solution in the sense of Definition 2.2 associated with a pair (ρ, μ) as above. With $\nu := \mu + \lambda$ (keep in mind that, relative to the pair (ρ, μ)), the concept does not depend on the measure absolutely continuously equivalent to $\mu + \lambda$), let $y_u(\cdot)$ be the associated λ-integrable selection of $t \mapsto F(t, u(t))$. Assume that μ is absolutely continuous with respect to λ. Then $\nu = \mu + \lambda$ is absolutely continuously equivalent to λ. As in the previous proof, one finds that the standard derivative $\dfrac{du}{dt}(\cdot)$ exists λ-almost everywhere and is a denstity in $L^1_\lambda(I, H)$ of du relative to λ, and further there exists some Borel set $B_1 \subset I$ with $\lambda(B_1) = 0$ and such that $\dfrac{du}{dt}(t) = \dfrac{du}{d\lambda}(t)$ for all $t \in I \setminus B_1$. From the absolute continuity of $\mu + \lambda$ with respect to λ and from (2.16) there exists a Borel subset $B_2 \subset I$ such that $\lambda(B_2) = 0$ and for all $t \in I \setminus B_2$

$$\frac{du}{d\lambda}(t) \text{ exist and } \quad \frac{du}{d\lambda}(t) + y_u(t)\frac{d\lambda}{d\lambda}(t) \in -N\big(C(t); u(t)\big).$$

It results that $\lambda(B_1 \cup B_2) = 0$ and

$$\frac{du}{dt}(t) + y_u(t) \in -N\big(C(t); u(t)\big), \quad \forall t \in I \setminus (B_1 \cup B_2).$$

The mapping $u(\cdot)$ is then a solution in the usual sense. ■

In view of the uniqueness, we state the following proposition concerning a particular chain rule for differential measures. Its statement is a consequence of a more general result from J.J. Moreau [55].

Proposition 3.3 *Let H be a Hilbert space, ν be a positive Radon measure on the closed bounded interval I, and $u(\cdot) : I \to H$ be a right continuous with bounded variation mapping such that the differential measure du has $\frac{du}{d\nu}$ as a density relative to ν. Then, the function $\Phi : I \to \mathbb{R}$ with $\Phi(t) := \|u(t)\|^2$ is a right continuous with bounded variation function whose differential measure $d\Phi$ satisfies, in the sense of the order of real measures,*

$$d\Phi \leq 2\langle u(\cdot), \frac{du}{d\nu}(\cdot)\rangle d\nu.$$

From the latter proposition the uniqueness of solution for (2.7) in the convex framework can then be established as in Moreau [55]. The proof is reproduced in order to keep the paper self-contained. In the proof and in the rest of the paper $\mathbf{1}_A$ denotes the function defined by $\mathbf{1}_A(t) = 1$ if $t \in A$ and $\mathbf{1}_A(t) = 0$ otherwise.

Proposition 3.4 *Assume that the closet sets $C(t)$ are convex. Let $u_1, u_2 : [T_0, T] \to H$ be right-continuous with bounded variation. If u_i (for $i = 1, 2$) is a solution of the differential inclusion $du \in -N\big(C(t); u(t)\big)$ in the sense of Definition 2.1 with initial condition $u_i(T_0) \in C(T_0)$ and associated pair (ρ_i, μ_i), then the function $t \mapsto \|u_1(t) - u_2(t)\|$ is non-increasing on the interval $I := [T_0, t]$.*

Consequently, with $u_0 \in C(T_0)$ the measure differential inclusion

$$du \in -N\big(C(t); u(t)\big) \quad \text{with initial condition } u(T_0) = u_0$$

has at most one solution in the sense of Definition 2.1.

Proof. Set $\mu := \mu_1 + \mu_2$ and note that each measure μ_i (with $i = 1, 2$) is absolutely continuous with respect to the Radon measure μ. By (2.17) the vector measure du_i is absolutely continuous with respect to μ and the mapping u_i is a solution associated to the pair (ρ_i, μ) of the differential inclusion $du \in -N\big(C(t); u(t)\big)$ with initial condition $u(T_0) = u_i(T_0)$. This yields that the differential measure du_i admits $\dfrac{du_i}{d\mu}(\cdot)$ as density relative to μ and

$$\frac{du_i}{d\mu}(t) \in -N\big(C(t); u_i(t)\big) \quad \text{for } \mu - \text{almost all } t \in I.$$

It ensues according to the definition of the normal cone of a convex set that, for μ-almost all $t \in I$

$$\left\langle \frac{du_1}{d\mu}(t) - \frac{du_2}{d\mu}(t), u_1(t) - u_2(t) \right\rangle \leq 0. \tag{3.1}$$

Using Proposition 3.3 the function $\Phi(\cdot) := \|u_1(\cdot) - u_2(\cdot)\|^2$ is right continuous with bounded variation and, with respect to the order of real measures,

$$d\Phi \leq \left\langle \frac{du_1}{d\mu}(\cdot) - \frac{du_2}{d\mu}(\cdot), u_1(\cdot) - u_2(\cdot) \right\rangle \mu \leq 0,$$

where the latter inequality is due to (3.1). For $s < t$ in I it results that

$$\Phi(t) - \Phi(s) = \int_I \mathbf{1}_{]s,t]}(\tau)\, d\Phi(\tau) \leq 0,$$

which confirms the non-increasing property of the function $\sqrt{\Phi(\cdot)}$.

The uniqueness with the initial condition $u_0 \in C(T_0)$ directly follows. ∎

As usual, in the next proposition and in the rest of the paper, for a mapping $u : I \to H$ we will denote by $u(t^-)$ the left-hand side limit of u at $t \in]T_0, T]$, that is, $u(t^-) := \lim_{\tau \uparrow t} u(\tau)$. When u is of bounded variation, $u(t^-)$ exists for all $t \in]T_0, T]$. We will also put by convention $u(T_0^-) = u(T_0)$.

Theorem 3.1 *Let $N(\cdot;\cdot)$ be the proximal normal cone and, with $I := [T_0, T]$, let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values. Let also $F : I \times H \rightrightarrows H$ be a set-valued mapping as above. Let $u : I \to H$ be a solution of* (2.15) *in the sense of Definition 2.2, associated with an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ satisfying* (2.8)*. Let $y_u(\cdot)$ be the associated λ-integrable selection of $t \mapsto F(t, u(t))$. Assume that $\|u(t)\| \leq \rho$ for all $t \in I := [T_0, T]$ (this being satisfied in particular whenever $\rho = +\infty$). Then, with $I_0 := \{t \in I : \mu(\{t\}) = 0\}$, given any positive Radon measure ν absolutely continuously equivalent to $\mu + \lambda$, for ν-a.e. $t \in I_0$ one has*

$$\left\| \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \right\| \leq \frac{d\mu}{d\nu}(t) + \|y_u(t)\| \frac{d\lambda}{d\nu}(t).$$

If in addition the solution u satisfies

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \textit{for all}\, t \in I,$$

then the latter inequality concerning the norm of $\frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t)$ holds true for ν-a.e. $t \in I$.

Proof. The inequality being clearly invariant with respect to absolutely continuously equivalent measures ν, it suffices to show it with $\nu := \mu + \lambda$. By definition and by (2.16) there exists a ν-null Borel set $B \subset I$ such that for all $t \in I \setminus B$ the inclusion in (ii) of Definition (2.2) is fulfilled. Fix any $t \neq T_0$ in $I \setminus B$ satisfying $\mu(\{t\}) = 0$. Clearly, one has $\nu(\{t\}) = \mu(\{t\}) = 0$. Put $\zeta(t) := \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t)$. By definition of proximal normal, there exists some real $\eta > 0$ such that for every real $0 \leq \sigma \leq \eta$

$$u(t) \in \operatorname{Proj}_{C(t)}\big(u(t) - \sigma\zeta(t)\big).$$

Since $\nu(]s, t]) \to \nu(\{t\}) = 0$ as $s \uparrow t$, we can fix some $s_0 \in I$ with $s_0 < t$ such that $0 < \nu(]s, t]) < \eta$ for all $s \in [s_0, t[$. Then for every $s \in [s_0, t[$, with $\sigma_s := \nu(]s, t])$ the above inclusion gives

$$\begin{aligned}\sigma_s \|\zeta(t)\| = d_{C(t)}\left(u(t) - \sigma_s\zeta(t)\right) &\leq d_{C(t)}(u(s)) + \|u(t) - u(s) - \sigma_s\zeta(t)\| \\ &\leq \mu(]s, t]) + \|u(t) - u(s) - \sigma_s\zeta(t)\|,\end{aligned}$$

where the latter inequality is due to the ρ-retraction inequality assumption of $C(\cdot)$ and to the inclusion $u(s) \in C(s) \cap \rho\mathbb{B}$. Dividing by $\sigma_s > 0$ yields

$$\left\| \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \right\| \leq \frac{\mu(]s, t])}{\nu(]s, t])} + \left\| \frac{u(t) - u(s)}{\nu(]s, t])} - \frac{du}{d\nu}(t) - y_u(t)\frac{d\lambda}{d\nu}(t) \right\|,$$

so passing to the limit as $s \uparrow t$ we obtain

$$\begin{aligned}\left\| \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \right\| &\leq \frac{d\mu}{d\nu}(t) + \left\| \frac{du}{d\nu}(t) - \frac{du}{d\nu}(t) - y_u(t)\frac{d\lambda}{d\nu}(t) \right\| \\ &= \frac{d\mu}{d\nu}(t) + \|y_u(t)\| \frac{d\lambda}{d\nu}(t).\end{aligned}$$

In the case where $\mu(\{T_0\}) = 0$ we have $\nu(\{T_0\}) = 0$, so it is enough to consider $B \cup \{T_0\}$ in place of B. The inequality in the first statement of the theorem is then justified.

Now assume that $\|u(t) - u(t^-)\| \le \mu(\{t\})$ for all $t \in I$. We only need to consider any $t \in I$ with $\mu(\{t\}) > 0$, so $\nu(\{t\}) = \mu(\{t\}) > 0$. For such a real t, using (2.13) the inequality $\|u(t) - u(t^-)\| \le \mu(\{t\})$ allows us to write

$$\left\| \frac{du}{d\nu}(t) \right\| = \left\| \frac{u(t) - u(t^-)}{\nu(\{t\})} \right\| \le \frac{\mu(\{t\})}{\nu(\{t\})} = \frac{d\mu}{d\nu}(t).$$

Since the inequality $\nu(\{t\}) > 0$ entails $\dfrac{d\lambda}{d\nu}(t) = 0$ (see (2.13)), from the latter inequality we see that we still have the inequality

$$\left\| \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \right\| \le \frac{d\mu}{d\nu}(t) + \|y_u(t)\| \, \frac{d\lambda}{d\nu}(t).$$

This finishes the proof. ■

The next corollary contains in particular the important case when $C(\cdot)$ enjoys the absolute continuity property of its truncated retraction.

Corollary 3.1 *Let $N(\cdot;\cdot)$ be the proximal normal cone and, with $I := [T_0, T]$, let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values. Let also $F : I \times H \rightrightarrows H$ be a set-valued mapping as above. Let $u : I \to H$ be a solution of* (2.15) *in the sense of Definition 2.2, associated with an extended real $\rho \ge \|u_0\|$ and a positive Radon measure μ satisfying* (2.8). *Let $y_u(\cdot)$ be the associated λ-integrable selection of $t \mapsto F(t, u(t))$. Assume that $\mu(\{t\}) = 0$ and $\|u(t)\| \le \rho$ for all $t \in I := [T_0, T]$ (this inequality being satisfied in particular whenever $\rho = +\infty$). Then, given any positive Radon measure ν absolutely continuously equivalent to $\mu + \lambda$, for ν-a.e. $t \in I$ one has*

$$\left\| \frac{du}{d\nu}(t) + y_u(t)\frac{d\lambda}{d\nu}(t) \right\| \le \frac{d\mu}{d\nu}(t) + \|y_u(t)\| \, \frac{d\lambda}{d\nu}(t).$$

In particular, if μ is absolutely continuous with respect to λ, so μ is defined by $\mu(]s, t]) = v(t) - v(s)$, where $v : I \to [0, +\infty[$ is a non-decreasing absolutely continuous function, then for Lebesgue almost all $t \in I$

$$\left\| \frac{du}{dt}(t) + y_u(t) \right\| \le \frac{dv}{dt}(t) + \|y_u(t)\|.$$

Proof. The first statement follows directly from the above theorem. Assume now that μ is absolutely continuous with respect to λ. Let v be a non-decreasing absolutely continuous function such that $\mu(]s, t]) = v(t) - v(s)$ for all $s, t \in I$ with $s \le t$. We note that the Lebesgue measure λ is absolutely continuously equivalent to $\mu + \lambda$. We can then apply the first statement with the measure λ to obtain a Borel set $Q \subset I$ with $\lambda(Q) = 0$ such that, for all $t \in I \setminus Q$

$$\left\| \frac{du}{dt}(t) + y_u(t) \right\| \le \frac{dv}{dt}(t) + \|y_u(t)\|.$$

■

Arguing in the same way as in Theorem 3.1 when the set-valued mapping F is null, we obtain the following proposition.

Proposition 3.5 *Let $N(\cdot;\cdot)$ be the proximal normal cone and, with $I := [T_0, T]$, let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed values. Let $u : I \to H$ be a solution of* (2.7) *in the sense of Definition 2.1, associated with an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ satisfying* (2.8)*. Assume that $\|u(t)\| \leq \rho$ for all $t \in I := [T_0, T]$ (this being satisfied in particular whenever $\rho = +\infty$). Then, with $I_0 := \{t \in I : \mu(\{t\}) = 0\}$, given any positive Radon measure ν absolutely continuously equivalent to μ, for ν-a.e. $t \in I_0$ one has*

$$\left\| \frac{du}{d\nu}(t) \right\| \leq \frac{d\mu}{d\nu}(t).$$

If in addition the solution u satisfies the condition

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \text{for all } t \in I,$$

then the above inequality concerning the norm of $\dfrac{du}{d\nu}(t)$ holds true for ν-a.e. $t \in I$.

Remark 3.1 Using, for the Fréchet subdifferential (see, e.g., [48] for the definition and properties), the well-known equality similar to the second equality in (2.5) and modifying in a suitable way the proof of Theorem 3.1 with $y_u(t) = 0$, one can see that the above proposition is still valid with the Fréchet normal cone instead of the proximal one. ■

The following theorem examines the case when the inclusion (2.14) is fulfilled with a measure which is not assumed to be absolutely continuously equivalent to the data measure μ.

Theorem 3.2 *Let $N(\cdot;\cdot)$ be the proximal normal cone and let be given an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ satisfying* (2.8)*. Let $u : I \to H$ be a right-continuous mapping of bounded variation on $I := [T_0, T]$ with $\|u(t)\| \leq \rho$ for all $t \in I$ (which is satisfied in particular whenever $\rho = +\infty$). Assume that there exists a positive Radon measure ν_0 with respect to which the differential measure du is absolutely continuous and such that*
(a) $u(t) \in C(t)$ for all $t \in I$;
(b) for ν_0-a.e. $t \in I$,

$$\frac{du}{d\nu_0}(t) \in -N\big(C(t); u(t)\big);$$

(c) $\|u(t) - u(t^-)\| \leq \mu(\{t\})$ for all $t \in I$.

Then u is a solution (associated to the pair (ρ, μ)) of the measure differential inclusion (2.7)*.*

Proof. I) The measure ν_0 being absolutely continuous with respect to the Radon measure $\nu := \mu + \nu_0$, by the Radon-Nikodým property of the Hilbert space H it ensues that $\frac{du}{d\nu}(\cdot)$ exists as a density of the differential measure du relative to the measure ν. Then, there exists some Borel set $N_1 \subset I$ with $\nu(N_1) = 0$ such that

$$\frac{du}{d\nu}(t) = \frac{du}{d\nu_0}(t)\frac{d\nu_0}{d\nu}(t) \quad \text{for all } t \in I \setminus \ N_1.$$

Modifying if necessary the density mapping $\frac{du}{d\nu_0}(\cdot)$ with the value 0 we may suppose that

$$\frac{du}{d\nu_0}(t) \in -N\big(C(t); u(t)\big) \quad \text{for all } t \in I.$$

Consequently, we can choose some Borel set $N_2 \supset N_1$ with $\nu(N_2) = 0$ such that for each $t \in I \setminus N_2$ (keeping in mind that $N\big(C(t); u(t)\big)$ is a cone)

$$\lim_{s \uparrow t} \frac{u(t) - u(s)}{\nu(]s,t])} = \frac{du}{d\nu}(t) \in -N\big(C(t); u(t)\big).$$

Further, since μ is absolutely continuous with respect to ν, we may suppose that $\frac{d\mu}{d\nu}(t)$ exists for all $t \in I \setminus N_2$ and that $\frac{d\mu}{d\nu}(\cdot)$ is a density of μ relative to the positive measure ν. Fix any $t \in I \setminus N_2$ with $\mu(\{t\}) = 0$, so u is continuous at t according to the assumption $\|u(t) - u(t^-)\| \leq \mu(\{t\})$. Suppose for a moment that $\frac{du}{d\nu}(t) \neq 0$ and put

$$\zeta(t) := \left\|\frac{du}{d\nu}(t)\right\|^{-1} \left\|\frac{du}{d\nu}(t)\right\|, \quad \text{so } \zeta(t) \in -\partial_P d_{C(t)}(u(t)), \tag{3.2}$$

where the inclusion is due to the second equality in (2.5).

II) By definition of proximal subdifferential, we can choose some real $\sigma > 0$ and some neighborhood U of $u(t)$ such that, for all $x \in U$

$$\langle \zeta(t), x - u(t)\rangle \leq d_{C(t)}(x) - d_{C(t)}(u(t)) + \sigma\|x - u(t)\|^2.$$

By the left continuity of $u(\cdot)$ at t, there is some $s_0 \in I$ with $s_0 < t$ such that $u(s) \in U$ for all $s \in]s_0, t[$. Combining this with the last inequality above we can write for every $s \in]s_0, t[$

$$\begin{aligned}\langle \zeta(t), u(s) - u(t)\rangle &\leq d_{C(t)}(u(s)) - d_{C(t)}(u(t)) + \sigma\|u(s) - u(t)\|^2 \\ &= d_{C(t)}(u(s)) + \sigma\|u(s) - u(t)\|^2 \\ &\leq \operatorname{exc}_\rho\big(C(s), C(t)\big) + \sigma\|u(s) - u(t)\|\end{aligned}$$

(the latter inequality being due to the inclusion $u(s) \in C(s) \cap \rho\mathbb{B}$ since $\|u(s)\| \leq \rho$ by assumption). It ensues that, for every $t \in]s_0, t[$

$$\langle \zeta(t), u(s) - u(t)\rangle \leq \mu(]s,t]) + \sigma\|u(s) - u(t)\|^2.$$

On the other hand, the existence of $\dfrac{du}{d\nu}(t)$ along with its non-nullity means that the limit

$$\lim_{s\uparrow t}\big(u(t)-u(s)\big)/\nu(]s,t])$$

exists and is non-null, so (recalling the convention $\frac{0}{0}=0$) we may and do suppose that the real $s_0<t$ is such that the inequality $\nu(]s,t])>0$ is satisfied for all $s\in]s_0,t[$. Consequently, for every $s\in]s_0,t[$ we derive that

$$\left\langle \zeta(t),\frac{u(t)-u(s)}{\nu(]s,t])}\right\rangle \le \frac{\mu(]s,t])}{\nu(]s,t])}+\sigma\left\|\frac{u(t)-u(s)}{\nu(]s,t])}\right\|\,\|u(t)-u(s)\|,$$

hence taking the limit as $s\uparrow t$ gives

$$\left\langle \zeta(t),\frac{du}{d\nu}(t)\right\rangle \le \frac{d\mu}{d\nu}(t)+\sigma\left\|\frac{du}{d\nu}(t)\right\|\,\|u(t)-u(t^-)\|,\ \textit{i.e.},\ \left\|\frac{du}{d\nu}(t)\right\|\le\frac{d\mu}{d\nu}(t),$$

where the latter inequality is due to the left-continuity of u at the chosen point t. Since the inequality obviously holds if $\dfrac{du}{d\nu}(t)=0$, it results that

$$\left\|\frac{du}{d\nu}(t)\right\|\le\frac{d\mu}{d\nu}(t)\quad\text{for all } t\in I\setminus N_2 \text{ with } \mu(\{t\})=0.$$

Take now $t\in I\setminus N_2$ with $\mu(\{t\})>0$, so $\nu(\{t\})>0$ too. In this case we can write by (2.13) and by the assumption (c)

$$\left\|\frac{du}{d\nu}(t)\right\|=\frac{\|u(t)-u(t^-)\|}{\nu(\{t\})}\le\frac{\mu(\{t\})}{\nu(\{t\})}=\frac{d\mu}{d\nu}(t).$$

All together we have proved that

$$\left\|\frac{du}{d\nu}(t)\right\|\le\frac{d\mu}{d\nu}(t)\quad\text{for all } t\in I\setminus N_2.$$

From this inequality and from the property that $\dfrac{du}{d\nu}(\cdot)$ and $\dfrac{d\mu}{d\nu}(\cdot)$ are densities relative to ν of the vector measure du and the real measure μ respectively, we see that the vector measure du is absolutely continuous with respect to the positive measure μ.

III) The space H being a Radon-Nikodým Banach space, the mapping $\dfrac{du}{d\mu}$ is well defined μ-almost everywhere and it is a density of the vector measure du relative to the measure μ. Taking into account the absolute continuity of the measure μ with respect to the measure ν we can choose some Borel set $N_3\supset N_2$ with $\nu(N_3)=0$ such that for every $t\in I\setminus N_3$ we have

$$\frac{du}{d\nu}(t)=\frac{du}{d\mu}(t)\frac{d\mu}{d\nu}(t),$$

which implies that

$$\frac{du}{d\mu}(t)\frac{d\mu}{d\nu}(t) \in -N\big(C(t);u(t)\big).$$

With $P := \{t \in I : \frac{d\mu}{d\nu}(t) = 0\}$, we have

$$\mu(P) = \int_P \frac{d\mu}{d\nu}(t)\,d\nu(t) = 0,$$

which gives, for $Q := P \cup N_3$

$$0 \le \mu(Q) \le \mu(P) + \nu(N_3) = 0.$$

Further, for every $t \in I \setminus Q$ clearly $\frac{d\mu}{d\nu}(t) > 0$, thus the last above inclusion involving the normal cone $N\big(C(t);u(t)\big)$ becomes

$$\frac{du}{d\mu}(t) \in -N\big(C(t);u(t)\big),$$

so $u(\cdot)$ is a solution (associated with the pair (ρ,μ)) of the measure differential inclusion (2.7). ■

Remark 3.2 The proof makes clear that the theorem is still valid with the Fréchet normal cone in place of the proximal one. ■

In view of the next theorem concerned with the situation of prox-regular sets, we will need the next lemma from [32] (see also [12]) for such sets. Recall first that, given an extended real $r \in]0,+\infty]$, a nonempty closed set S of a Hilbert space H is *r-prox-regular* if and only if the distance function d_S is Fréchet differentiable on the r-open tube $U_r(S)$ defined by

$$U_r(S) := \{x \in H \,:\, 0 < d_S(x) < r\}.$$

It is known (see, e.g., [64]) that this is equivalent to the single-valuedness and continuity of the metric projection Proj_S on $U_r(S)$. Many other properties of the normal cone of S are known (see, e.g., [64, 27] and the references therein) as verifiable criteria (that is, necessary and sufficient conditions) for the r-prox-regularity of the closed set S. For example, S is r-prox-regular (see [64]) if and only if the following hypomonotonicity property for the normal cone $N^P(S;\cdot)$ holds true

$$\langle \xi_1 - \xi_2, x_1 - x_2\rangle \ge -\frac{1}{r}\|x_1 - x_2\|^2$$

for all $x_i \in S$ and $\xi_i \in N^P(S;x_i) \cap \mathbb{B}$ with $i = 1,2$. Furthermore, all the normal cones of a prox-regular set S coincide, and at any point x with $d_S(x) < r$ all the subdifferentials of the distance function d_S coincide as well. It is also worth recalling that any closed convex set of the Hilbert space H is r-prox-regular with $r = +\infty$. For operations on prox-regular sets, we refer to [5, 27].

Lemma 3.1 *Let S be an r-prox-regular set of the Hilbert space H with $r \in]0, +\infty]$. Let $x \in S$ and $\xi \in \partial d_S(x)$. Then, for any $y \in H$ with $d_S(y) < r$, one has*

$$\langle \xi, y - x \rangle \leq \frac{2}{r}\|x - y\|^2 + d_S(y).$$

Theorem 3.3 *Let $N(\cdot;\cdot)$ be the proximal normal cone and let be given an extended real ρ and a positive Radon measure μ satisfying* (2.8)*. Assume that all the sets $C(t)$ (with $t \in I := [T_0, t]$) are r-prox-regular for some $r \in]0, +\infty]$ and that $\mu(\{t\}) < r$ for all $t \in I$. Let $u : I \to H$ be a right-continuous mapping of bounded variation with $\|u(t)\| \leq \rho$ for all $t \in I$. Assume that there exists a Radon measure ν_0 with respect to which the differential measure du is absolutely continuous and such that*
(a) $u(t) \in C(t)$ for all $t \in I$;
(b) for ν_0-a.e. $t \in I$,

$$\frac{du}{d\nu_0}(t) \in -N(C(t); u(t));$$

(c) for some $0 \leq \gamma < 1$,

$$\|u(t) - u(t^-)\| \leq (\gamma r)/2 \quad \textit{for all}\, t \in I.$$

Then, $u(\cdot)$ is a solution (associated with the pair (ρ, μ)) of the measure differential inclusion (2.7)*.*

Proof. Consider the Radon measure $\nu := \mu + \nu_0$. We proceed as in Step I in the proof of Theorem 3.2 and we obtain $\zeta(t) \in \partial_P d_{C(t)}\big(u(t)\big)$. For any $s \in I$ with $s < t$ the assumption of bounded retraction of $C(\cdot)$ along ρ-truncation (2.8) and the assumption $\|u(s)\| \leq \rho$ guarantee that

$$d_{C(t)}(u(s)) \leq \mathrm{exc}_\rho\big(C(s), C(t)\big) \leq \mu(]s, t]),$$

hence $d_{C(t)}\big(u(t^-)\big) \leq \mu(\{t\}) < r$. Then, we can choose some $s_0 < t$ in I such that $d_{C(t)}(u(s)) < r$ for all $s \in]s_0, t[$. From this and Lemma 3.1 it ensues, for every $s \in]s_0, t[$

$$\begin{aligned}\langle \zeta(t), u(s) - u(t) \rangle &\leq d_{C(t)}(u(s)) - d_{C(t)}(u(t)) + \frac{2}{r}\|u(s) - u(t)\|^2 \\ &= d_{C(t)}(u(s)) + \frac{2}{r}\|u(s) - u(t)\|^2 \\ &\leq \mu(]s, t]) + \frac{2}{r}\|u(s) - u(t)\|^2.\end{aligned}$$

Taking into account the existence of the limit $\lim\limits_{s \uparrow t} \big(u(t) - u(s)\big)/\nu(]s, t])$ as well as its non-nullity, we may suppose that the real $s_0 < t$ satisfies that $\nu(]s, t]) > 0$ for all $s \in]s_0, t[$. It results that, for every $s \in]s_0, t[$

$$\left\langle \zeta(t), \frac{u(t) - u(s)}{\nu(]s, t])} \right\rangle \leq \frac{\mu(]s, t])}{\nu(]s, t])} + \frac{2}{r}\left\| \frac{u(t) - u(s)}{\nu(]s, t])} \right\| \|u(t) - u(s)\|,$$

hence taking the limit as $s \uparrow t$ gives

$$\left\langle \zeta(t), \frac{du}{d\nu}(t) \right\rangle \leq \frac{d\mu}{d\nu}(t) + \frac{2}{r} \left\| \frac{du}{d\nu}(t) \right\| \|u(t) - u(t^-)\|$$
$$\leq \frac{d\mu}{d\nu}(t) + \gamma \left\| \frac{du}{d\nu}(t) \right\|.$$

This entails that

$$(1-\gamma) \left\| \frac{du}{d\nu}(t) \right\| \leq \frac{d\mu}{d\nu}(t).$$

Since the inequality obviously holds if $\frac{du}{d\nu}(t) = 0$, we deduce that

$$(1-\gamma) \left\| \frac{du}{d\nu}(t) \right\| \leq \frac{d\mu}{d\nu}(t) \quad \text{for all } t \in I \setminus N_2.$$

As in the end of step II of the proof of Theorem 3.1, from the latter inequality and from the fact that $\frac{du}{d\nu}(\cdot)$ and $\frac{d\mu}{d\nu}(\cdot)$ are densities relative to ν of the vector measure du and the real measure μ respectively, we derive that the vector measure du is absolutely continuous with respect to the positive measure μ.

We can then apply Step III of the proof of Theorem 3.1 to finish the proof. ■

Keeping in mind that closed convex sets of the Hilbert space H are r-prox-regular with $r = +\infty$, the following corollary follows from the above theorem.

Corollary 3.2 *Let be given an extended real $\rho \geq \|u_0\|$ and a positive Radon measure μ satisfying* (2.8)*. Assume that all the closed sets $C(t)$ (with $t \in I := [T_0, T]$) are convex. Let $u : I \to H$ be a right-continuous mapping of bounded variation satisfying $\|u(t)\| \leq \rho$ for all $t \in I$. The following assertions are equivalent:*

1. *The mapping u is a solution associated to the pair (ρ, μ) of the measure differential inclusion* (2.7)*;*

2. *there exists a Radon measure ν with respect to which the differential measure du is absolutely continuous and such that the following conditions (a) and (b) hold:*

(a) $u(t) \in C(t)$ for all $t \in I$;

(b) for ν-a.e. $t \in I$,

$$\frac{du}{d\nu}(t) \in -N\big(C(t); u(t)\big).$$

3. *the above condition (a) holds as well as the condition (b'):*

(b') for $|du|$-a.e. $t \in I$,

$$\frac{du}{|du|}(t) \in -N\big(C(t); u(t)\big).$$

Proof. By the preceding corollary we know that the assertions **1**. and **2**. are equivalent. Since the vector measure du is absolutely continuous with respect to its variation measure $|du|$, the above equivalence tells us that the third assertion implies the first.

Suppose now that u is a solution, so du is absolutely continuous with respect to μ and there is a Borel set N of I with $\mu(N) = 0$ such that

$$\frac{du}{d\mu}(t) \in -N\big(C(t); u(t)\big) \quad \text{for all } t \in I \setminus N.$$

Then the variation measure $|du|$ of du is absolutely continuous with respect to μ, and since du is absolutely continuous with respect to its variation measure $|du|$, there exists a Borel set $N_1 \supset N$ with $\mu(N_1) = 0$ such that

$$\frac{du}{d\mu}(t) = \frac{du}{|du|}(t)\frac{d(|du|)}{d\mu}(t) \quad \text{for all } t \in I \setminus N_1.$$

Set $P := \{t \in I : \dfrac{d(|du|)}{d\mu}(t) = 0\}$ and note by the absolute continuuity of $|du|$ with respect to μ that

$$|du|(P) = \int_P \frac{d(|du|)}{d\mu}\, d\mu(t) = 0.$$

Noting also that $|du|(N_1) = 0$ since $\mu(N_1) = 0$ and $|du|$ is absolutely continuous with respect to μ, we obtain that $|du|(N_1 \cup P) = 0$. On the other hand, from what precedes and from the conical property of the normal cone we see that, for all $t \in I \setminus (N_1 \cup P)$

$$\frac{du}{|du|}(t) \in -N\big(C(t); u(t)\big).$$

This justifies that the first assertion entails the third, and finishes the proof. ■

Taking $\rho = +\infty$ in the previous corollary immediately yields this other corollary:

Corollary 3.3 *Assume that all the closed sets $C(t)$ in H are convex and let be given a positive Radon measure μ on $I = [T_0, T]$ such that*

$$\operatorname{exc}\big(C(s), C(t)\big) \le \mu(]s,t]) \quad \textit{for all } s,t \in I \textit{ with } s \le t.$$

Let $u : I \to H$ be a right-continuous mapping of bounded variation. The following three assertions are pairwise equivalent:
1. *The mapping u is a solution of the measure differential inclusion*

$$\begin{cases} du \in -N\big(C(t); u(t)\big) \\ u(T_0) = u_0 \in C(T_0); \end{cases}$$

2. *there exists a Radon measure ν on I with respect to which the differential measure du is absolutely continuous and such that the following conditions (a)*

and (b) hold:
(a) $u(t) \in C(t)$ *for all* $t \in I$*;*
(b) for ν*-a.e.* $t \in I$*,*

$$\frac{du}{d\nu}(t) \in -N\big(C(t); u(t)\big);$$

3. *the above condition (a) holds as well as the condition (b'):*
(b') for $|du|$*-a.e.* $t \in I$*,*

$$\frac{du}{|du|}(t) \in -N\big(C(t); u(t)\big).$$

The latter corollary translates that for a solution, with closed convex sets $C(t)$ and under the (usual) bounded retraction of the set-valued mapping $C(\cdot)$, there is no privilege of the property of absolute continuity equivalence to μ.

With prox-regular sets a uniqueness result in the line of Proposition 3.4 can also be established. Recall first a Gronwall-type lemma from [47].

Lemma 3.2 *Let* I *be an interval in* $\mathbb{R}$ *and* $T_0 \in I$*. Let* ν *be a positive Radon measure on* I*, and* $g, \varphi : I \to [0, +\infty[$ *be two functions such that* $g \in L^1_{\text{loc}}(I, \mathbb{R}, \nu)$ *and* $\varphi \in L^\infty_{\text{loc}}(I, \mathbb{R}, \nu)$*. Assume that:*
(i) for some fixed $\theta \in [0, +\infty[$

$$0 \le g(t)\nu(\{t\}) \le \theta < 1 \quad \textit{for all}\, t \in]T_0, T];$$

(ii) for some fixed $\alpha \in [0, +\infty[$

$$\varphi(t) \le \alpha + \int_{]T_0, t]} g(s)\varphi(s)\, d\nu(s) \quad \textit{for all}\, t \ge T_0.$$

Then, for all $t \in]T_0, T]$ *one has the inequality*

$$\varphi(t) \le \alpha \,\exp\left(\frac{1}{1-\theta}\int_{]T_0, t]} g(s)\, d\nu(s)\right).$$

Proposition 3.6 *Assume that, for some* $r \in]0, +\infty]$*, all the closed sets* $C(t)$ *are* r*-prox-regular for* $t \in I := [T_0, T]$*. Let* $f : I \times H \to H$ *be a mapping Borel-measurable in* t *and such that, for each bounded subset* B *of* H *there exists a non-negative function* $\ell_B \in L^1(I, \lambda)$ *for which the hypomonotonicity property*

$$\langle f(t, x_1) - f(t, x_2), x_1 - x_2 \rangle \ge -\ell_B(t)\|x_1 - x_2\|^2$$

is satisfied (as holds under the $\ell_B(t)$*-Lipschitz property of* $f(t, \cdot)$ *on* B*).*

Then the measure differential inclusion (2.15) *with* f *in place of* F *and with* $u_0 \in C(T_0)$ *as initial condition has at most one (right-continuous with bounded variation solution)* $u(\cdot)$ *satisfying*

$$\sup_{t \in]T_0, T]} \|u(t) - u(t^-)\| < \frac{r}{2}.$$

Proof. Let $u_1(\cdot)$ and $u_2(\cdot)$ be two solutions such that, for each $i = 1, 2$

$$\sup_{t\in]T_0,T]} \|u_i(t) - u_i(t^-)\| < \frac{r}{2}.$$

Let (ρ_i, μ_i) the associated pair for u_i and put $\nu := \mu_1 + \mu_2 + \lambda$. For each $i = 1, 2$ by (2.16) we have, for ν-almost all $t \in I$

$$\frac{du_i}{d\nu}(t) + f(t, u_i(t))\frac{d\lambda}{d\nu}(t) \in -N\big(C(t); u_i(t)\big).$$

Let B be a bounded set containing $u_1(I)$ and $u_2(I)$. Considering the non-negative function $g \in L^1(I, \mathbb{R}, \nu)$ given by

$$g(t) := 2\left[\frac{d\lambda}{d\nu}(t)\ell_B(t) + \frac{1}{2r}\left(\sum_{i=1}^{2}\Big(\|\frac{du_i}{d\nu}(t)\| + \|f(t, u_i(t))\|\,\frac{d\lambda}{d\nu}(t)\Big)\right)\right],$$

and proceeding as in [4, Theorem 6.1] we obtain that, for all $t \in I$

$$\|u_1(t) - u_2(t)\|^2 \le \int_{]T_0,T]} g(s)\,\|u_1(s) - u_2(s)\|^2\, d\nu(s).$$

Further, with $\gamma := 2\max_{i\in\{1,2\}}\sup_{s\in]T_0,T]}\|u_i(s) - u_i(s^-)\| < r$ we also see (as in [4, Theorem 6.1]) that, for ν-almost every $t \in I$

$$0 \le g(t)\nu(\{t\}) \le \gamma/r < 1.$$

From this and Lemma 3.2 it results that $\|u_1(t) - u_2(t)\|^2 \le 0$, which finishes the proof. ■

4 Existence theorem

The following lemma from Moreau [56, Lemma 1(2a)] will be used in the proof of the existence theorem below.

Lemma 4.1 *Let S be a nonempty closed convex set of the Hilbert space H. Then, for all $x, y \in H$ one has*

$$\|x - \operatorname{proj}(y, S)\|^2 - \|x - y\|^2 \le 2d(x, S)\,d(y, S).$$

A fundamental stability property of the directional derivative of the distance function from a moving convex set will also be needed.

Proposition 4.1 *Let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed convex subsets of the Hilbert space H. Assume that there exists a Radon measure μ on I such that, for some $\rho \in]0, +\infty]$*

$$\operatorname{exc}_\rho\big(C(s), C(t)\big) \le \mu(]s, t]) \quad \text{for all } s, t \in I \text{ with } s \le t.$$

Let $\bar{t} \in I$ and $\bar{x} \in C(\bar{t})$, and let $(t_n)_n$ in I with $t_n \geq \bar{t}$ and $\mu(]\bar{t}, t_n]) \underset{n\to\infty}{\longrightarrow} 0$, and let $(x_n)_n$ converging to $\bar{x}$ with $x_n \in C(t_n)$. If $\|\bar{x}\| < \rho$, then for every $h \in H$

$$\limsup_{n\to\infty} d'_{C(t_n)}(x_n; h) \leq d'_{C(\bar{t})}(\bar{x}; h).$$

Proof. Since $\lim_{x\to\bar{x}} \operatorname{proj}(x, C(\bar{t})) = \bar{x}$, for x near enough to $\bar{x}$ we note that $\|\operatorname{proj}(x, C(\bar{t}))\| < \rho$, which entails for such elements x

$$\operatorname{proj}\big(x, C(\bar{t})\big) = \operatorname{proj}\big(x, C(\bar{t}) \cap \rho\mathbb{B}\big), \quad \text{hence } d\big(x, C(\bar{t}) \cap \rho\mathbb{B}\big) = d\big(x, C(\bar{t})\big).$$

Let $(t_n)_n$ and $(x_n)_n$ be as in the statement. Fix any $h \in H$. Then, for each real $\tau > 0$ small enough, by (2.6) we have for all n

$$\begin{aligned} d'_{C(t_n)}(x_n; h) &\leq \tau^{-1} d_{C(t_n)}(x_n + \tau h) \leq \tau^{-1}\Big(\|x_n - \bar{x}\| + d_{C(t_n)}(\bar{x} + \tau h)\Big) \\ &\leq \tau^{-1}\Big(\|x_n - \bar{x}\| + d\big(\bar{x} + \tau h, C(\bar{t}) \cap \rho\mathbb{B}\big) + \mu(]\bar{t}, t_n])\Big) \\ &= \tau^{-1}\Big(\|x_n - \bar{x}\| + d_{C(\bar{t})}(\bar{x} + \tau h) + \mu(]\bar{t}, t_n])\Big), \end{aligned}$$

where the first inequality is due to (2.3) and the third to (2.6). It follows that, for $\tau > 0$ small enough

$$\limsup_{n\to\infty} d'_{C(t_n)}(x_n; h) \leq \tau^{-1} d_{C(\bar{t})}(\bar{x} + \tau h) = \tau^{-1}\Big(d_{C(\bar{t})}(\bar{x} + \tau h) - d_{C(\bar{t})}(\bar{x})\Big),$$

which ensures the desired inequality

$$\limsup_{n\to\infty} d'_{C(t_n)}(x_n; h) \leq d'_{C(\bar{t})}(\bar{x}; h).$$

■

We can now prove, *using some ideas from Moreau* [56], the theorem concerning the above measure differential inclusion. The case of prox-regular sets $C(t)$ will be studied elsewhere.

Given a sequence of positive reals $(\varepsilon_n)_{n\geq 1}$ with $\varepsilon_n \downarrow 0$, we will have to consider in the proof, for each $n \geq 1$ a sequence $(\varepsilon_{n,j})_{j\geq 0}$ such that

$$\varepsilon_{n,j} < \varepsilon_n, \quad \sum_{j=0}^{\infty} \varepsilon_{n,j} = +\infty \quad \text{and} \quad \sum_{j=0}^{\infty} (\varepsilon_{n,j})^2 \leq 1/n.$$

Setting $S := \sum_{j=0}^{\infty} \frac{1}{(j+1)^2}$, such a sequence $(\varepsilon_{n,j})_{j\geq 0}$ is obtained, for example, with $\varepsilon_{n,j} := \min\{\frac{1}{(j+1)\sqrt{S}\sqrt{n}}, \frac{\varepsilon_n}{2}\}$.

Theorem 4.1 *Let $C(\cdot) : [T_0, T] \rightrightarrows H$ be a set-valued mapping from $[T_0, T]$ into the nonempty closed convex subsets of the Hilbert space H, and let $u_0 \in C(T_0)$.*

Assume that there exist a positive Radon measure μ on $I := [T_0, T]$, a real $\rho_0 \geq \|u_0\|$ and an extended real $\rho > \rho_0$ such that for all $s, t \in I$ with $s \leq t$

$$\mathrm{exc}_\rho\big(C(s), C(t)\big) \leq \mu(]s,t]),$$

and such that for all $t_1 < \cdots < t_k$ in I the inequality

$$\|\big(\mathrm{proj}_{\,C(t_k)} \circ \cdots \circ \mathrm{proj}_{\,C(t_1)}\big)(u_0)\| \leq \rho_0$$

is satisfied.

Then, the following sweeping process

$$\begin{cases} du \in -N\big(C(t); u(t)\big) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right continuous with bounded variation solution.

Furthermore, the solution $u(\cdot)$ satisfies the inequality

$$\|u(t) - u(s)\| \leq \mu(]s,t]) \quad \textit{for all } s \leq t \textit{ in } I,$$

and for all $t \in I$

$$\|u(t)\| \leq \min\{\rho_0, \|u_0\| + \mu(]T_0, T])\}.$$

Proof. The uniqueness follows from Proposition 3.4.

Let us prove the existence.

If $\mu(]T_0, T]) = 0$, then $C(T_0) \cap \rho\mathbb{B} \subset C(t)$ for all $t \in I$, so the constant mapping $u(\cdot)$ with $u(t) := u_0$ for all $t \in I$ is the solution.

From now on, we suppose that $\mu(]T_0, T]) > 0$. As in Moreau [56] (see also Castaing and Monteiro Marques [20]), consider the function $v(\cdot) : I \to \mathbb{R}$ defined by

$$v(t) := \mu(]T_0, t])$$

and set

$$V := v(T) = \mu(]T_0, T]).$$

The function $v(\cdot)$ is non-decreasing and right continuous with $v(T_0) = 0$. Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive reals with $\varepsilon_n < \mu(]T_0, T])$ and $\varepsilon_n \downarrow 0$. For each integer $n \geq 1$, fix a sequence of positive reals $(\varepsilon_{n,j})_{j \geq 0}$ such that

$$\varepsilon_{n,j} < \varepsilon_n, \quad \sum_{j=0}^{\infty} \varepsilon_{n,j} = +\infty \quad \text{and} \quad \sum_{j=0}^{\infty} (\varepsilon_{n,j})^2 \leq 1/n. \tag{4.1}$$

Let, for each $n \in \mathbb{N}$, a partition $0 = V_0^n < V_1^n < \cdots < V_{q_n}^n = V$ such that

$$V_{j+1}^n - V_j^n \leq \varepsilon_{n,j} < \varepsilon_n, \ \forall j = 0, \cdots, q_n - 1, \tag{4.2}$$

$$\text{and} \quad \{V_0^n, \cdots, V_{q_n}^n\} \subset \{V_0^{n+1}, \cdots, V_{q_{n+1}}^{n+1}\}.$$

Put $V^n_{1+q_n} := V + \varepsilon_{n,q_n}$. For each $n \in \mathbb{N}$, consider the partition of I associated with the subsets

$$J^n_j := v^{-1}([V^n_j, V^n_{j+1}[), \ j = 0, 1, \cdots, q_n,$$

and note that $(J^m_j)^{q_m}_{j=0}$ is a refinement of $(J^n_j)^{q_n}_{j=0}$ whenever $m \geq n$. Since $v(\cdot)$ is non-decreasing and right continuous, it is easy to see that, for each $j = 0, 1, \cdots, q_n - 1$, the set J^n_j is either empty or an interval of the form $[r, s[$ with $r < s$. Furthermore, we have either $J^n_{q_n} = \{T\}$ or $J^n_{q_n} = [r, T]$ for some $r < T$. All together produce an integer $p(n) \in \mathbb{N}$ and a finite sequence

$$T_0 = t^n_0 < t^n_1 < \cdots < t^n_{p(n)} = T$$

such that, for each $i \in \{0, \cdots, p(n) - 1\}$, there is some $j_n(i) \in \{0, \cdots, q_n\}$ for which $[t^n_i, t^n_{i+1}[= J^n_{j_n(i)}$ if $i < p(n) - 1$ and $[t^n_i, t^n_{i+1}[= J^n_{j_n(i)} \setminus \{T\}$ if $i = p(n) - 1$. It ensues that

$$i \neq i' \implies j_n(i) \neq j_n(i') \tag{4.3}$$

and that by (4.2), for any $i \in \{0, \cdots, p(n) - 1\}$,

$$\mu(]t^n_i, t]) = v(t) - v(t^n_i) \leq \varepsilon_{n,j_n(i)} \leq \varepsilon_n \quad \text{for all } t \in [t^n_i, t^n_{i+1}[,$$

which entails in particular

$$\mu(]t^n_i, t^n_{i+1}[) \leq \varepsilon_{n,j_n(i)} \leq \varepsilon_n. \tag{4.4}$$

Now, put $u^n_0 := u_0$ and $u^n_1 := \operatorname{proj}_{C(t^n_1)}(u^n_0)$, and define by induction $\{u^n_i : i = 0, \cdots, p(n)\}$ such that

$$u^n_{i+1} := \operatorname{proj}_{C(t^n_{i+1})}(u^n_i), \quad \text{for all } i = 0, \cdots, p(n) - 1. \tag{4.5}$$

By assumption we have

$$\|u^n_i\| \leq \rho_0 \quad \text{for all } i \in \{0, \cdots, p(n) - 1\}. \tag{4.6}$$

Taking (4.5) into account, we derive from the inequality $\|u_0\| \leq \rho$ and from the retraction assumption of $C(\cdot)$ along ρ-truncation that

$$\|u^n_1 - u^n_0\| = d_{C(t^n_1)}(u^n_0) \leq \operatorname{exc}_\rho\big(C(t^n_0), C(t^n_1)\big) \leq \mu(]T_0, t^n_1]).$$

By the inequality $\|u^n_1\| \leq \rho_0$, the retraction of $C(\cdot)$ along ρ-truncation again gives

$$\|u^n_2 - u^n_1\| = d_{C(t^n_2)}(u^n_1) \leq \operatorname{exc}_\rho\big(C(t^n_1), C(t^n_2)\big) \leq \mu(]t^n_1, t^n_2]).$$

We then obtain by induction for all $i = 0, 1, \cdots, p(n) - 1$

$$\|u^n_{i+1} - u^n_i\| \leq \mu(]t^n_i, t^n_{i+1}]) \leq \varepsilon_{n,j_n(i)} \leq \varepsilon_n. \tag{4.7}$$

The rest of the proof is divided in several steps.

Step 1. Construction of a Cauchy sequence of step mappings $(U_n(\cdot))_n$.
For each integer $n \geq 1$ define the step mapping $U_n : [T_0, T] \to H$ by $U_n(T) = u^n_{p(n)}$ and

$$U_n(t) = u^n_i \quad \text{if } t \in [t^n_i, t^n_{i+1}[\text{ with } i \in \{0, \cdots, p(n) - 1\},$$

and observe that $U_n(\cdot)$ is right continuous (and of course with bounded variation). In order to prove the Cauchy property of that sequence, fix any real $\eta > 0$ and choose an integer $N > 8/\eta^2$. Take any integer $n > N$. Fix any $i \in \{0, \cdots, p(N) - 1\}$. Let

$$t^n_k < \cdots < t^n_{k+k_n} < t^n_{k+k_n+1} \leq t^N_{i+2}$$

with $t^n_k = t^N_i$ and $t^n_{k+k_n} = t^N_{i+1}$. Denote by $x^N_i := U_N(t^N_i)$ the constant value of $U_N(\cdot)$ on $[t^N_i, t^N_{i+1}[$ and denote, for each $j = k, \cdots, k + k_n$, by

$$y^n_j := U_n(t^n_j) \quad \text{the constant value of } U_n(\cdot) \text{ on } [t^n_j, t^n_{j+1}[.$$

Then, for each $j = k, \cdots, k + k_n - 1$ we see that $y^n_{j+1} = \operatorname{proj}\big(y^n_j, C(t^n_{j+1})\big)$, hence Lemma 4.1 yields

$$\|x^N_i - y^n_{j+1}\|^2 - \|x^N_i - y^n_j\|^2 \leq 2d\big(x^N_i, C(t^n_{j+1})\big)d\big(y^n_j, C(t^n_{j+1})\big).$$

It ensues that, for any $p = 0, \cdots, k_n$

$$\|x^N_i - y^n_{k+p}\|^2 - \|x^N_i - y^n_k\|^2 \leq 2 \sum_{j=k}^{k+p-1} d\big(x^N_i, C(t^n_{j+1})\big)d\big(y^n_j, C(t^n_{j+1})\big), \tag{4.8}$$

with the convention that the latter sum and the similar ones below are zero for $p = 0$. Since $x^N_i \in C(t^N_i) = C(t^n_k)$ (keep in mind that $t^n_k = t^N_i$), we can also write by (4.6) and by the assumption of retraction along ρ-truncation, for every $j \in \{k, \cdots, k + k_n - 1\}$

$$d\big(x^N_i, C(t^n_{j+1})\big) \leq \operatorname{exc}_\rho\big(C(t^n_k), C(t^n_{j+1})\big) \leq \mu(]t^n_k, t^n_{j+1}]) \leq \mu(]t^N_i, t^N_{i+1}]).$$

From this and the inequality $\|y^n_j\| \leq \rho_0$ (see (4.6)) we deduce, for any $p = 0, 1, \cdots, k_n - 1$

$$\begin{aligned}
\sum_{j=k}^{k+p-1} d\big(x^N_i, C(t^n_{j+1})\big)d\big(y^n_j, C(t^n_{j+1})\big) &\leq \mu(]t^N_i, t^N_{i+1}]) \sum_{j=k}^{k+p-1} d\big(y^n_j, C(t^n_{j+1})\big) \\
&\leq \mu(]t^N_i, t^N_{i+1}]) \sum_{j=k}^{k+p-1} \operatorname{exc}_\rho\big(C(t^n_j), C(t^n_{j+1})\big) \\
&\leq \big(\mu(]t^N_i, t^N_{i+1}])\big)^2,
\end{aligned}$$

where the latter inequality is due again to the assumption of retraction along ρ-truncation. Taking any $t \in [t^N_i, t^N_{i+1}[$, this and (4.8) ensure that

$$\|U_N(t) - U_n(t)\|^2 - \|U_N(t^N_i) - U_n(t^N_i)\|^2 \leq 2\big(\mu(]t^N_i, t^N_{i+1}])\big)^2 \leq 2(\varepsilon_{N,j_N(i)})^2. \tag{4.9}$$

Keeping in mind that $t^N_{i+1} = t^n_{k+k_n}$, we note that $C(t^N_{i+1}) = C(t^n_{k+k_n})$ and that

$$\begin{aligned}
&\|U_N(t^N_{i+1}) - U_n(t^N_{i+1})\| \\
&= \|\text{proj}\,\big(U_N(t^N_i), C(t^N_{i+1})\big) - \text{proj}\,\big(U_n(t^n_{k+k_n-1}), C(t^n_{k+k_n})\big)\| \\
&= \|\text{proj}\,\big(U_N(t^N_i), C(t^N_{i+1})\big) - \text{proj}\,\big(U_n(t^n_{k+k_n-1}), C(t^N_{i+1})\big)\| \\
&\le \|U_N(t^N_i) - U_n(t^n_{k+k_n-1})\| = \|U_N(t^n_{k+k_n-1}) - U_n(t^n_{k+k_n-1})\|,
\end{aligned}$$

where the latter equality follows from the constant property of $U_N(\cdot)$ on $[t^N_i, t^N_{i+1}[$ and from the inclusion $t^n_{k+k_n-1} \in [t^N_i, t^N_{i+1}[$. Combining this with (4.9) and using the inclusion $t^n_{k+k_n-1} \in [t^N_i, t^N_{i+1}[$ again, we derive that

$$\begin{aligned}
&\|U_N(t^N_{i+1}) - U_n(t^N_{i+1})\|^2 - \|U_N(t^N_i) - U_n(t^N_i)\|^2 \\
&\le \|U_N(t^n_{k+k_n-1}) - U_n(t^n_{k+k_n-1})\|^2 - \|U_N(t^N_i) - U_n(t^N_i)\|^2 \le 2(\varepsilon_{N,j_N(i)})^2.
\end{aligned}$$

Iterating this argument we obtain with $\ell = 1, \cdots, i+1$

$$\|U_N(t^N_\ell) - U_n(t^N_\ell)\|^2 - \|U_N(t^N_{\ell-1}) - U_n(t^N_{\ell-1})\|^2 \le 2(\varepsilon_{N,j_N(\ell-1)})^2.$$

This and (4.9) furnish for all $t \in [t^N_i, t^N_{i+1}]$ the inequality

$$\begin{aligned}
\|U_N(t) - U_n(t)\|^2 &= \|U_N(t) - U_n(t)\|^2 - \|U_N(t^N_0) - U_n(t^N_0)\|^2 \\
&\le 2\sum_{\ell=0}^{\ell=i+1} (\varepsilon_{N,j_N(\ell)})^2 \le 2/N < \eta^2/4,
\end{aligned}$$

where the second inequality is due to (4.1) and (4.3). It then follows, for any $n \ge N$ and any $m \ge N$

$$\|U_n(t) - U_m(t)\| \le \eta \quad \text{for all } t \in [T_0, T],$$

which justifies the Cauchy property of the sequence $(U_n(\cdot))_n$ with respect to the norm of uniform convergence, so this sequence converges uniformly on $[T_0, T]$ to a mapping $u(\cdot)$, and (4.6) and the definition of $U_n(\cdot)$ entail that $\|u(t)\| \le \rho_0$ for all $t \in I$.

Step 2. Construction of another sequence $(u_n(\cdot))_n$ of right continuous mappings with bounded variation.

First, we note in the case when $\mu(]t^n_i, t^n_{i+1}]) = 0$, that $\text{ex}_\rho\big(C(t^n_i), C(t^n_{i+1})\big) = 0$, which means that $C(t^n_i) \cap \rho\mathbb{B} \subset C(t^n_{i+1})$; in such a case, it then results, according to the inequality $\|u^n_i\| \le \rho_0$ (see (4.6)), that

$$u^n_i \in C(t^n_i) \cap \rho\mathbb{B} \subset C(t^n_{i+1}), \quad \text{so } u^n_i = u^n_{i+1}, \tag{4.10}$$

keeping in mind that $u^n_{i+1} = \text{proj}\,\big(u^n_i, C(t^n_{i+1})\big)$.

Following [56] (see also [20]), define the mapping $u_n(\cdot) : I \to H$ by $u_n(T) := u^n_{p(n)}$ and if $\mu(]t^n_i, t^n_{i+1}]) = 0$

$$u_n(t) = u^n_i \quad \text{for all } t \in [t^n_i, t^n_{i+1}[,$$

if $\mu(]t_i^n, t_{i+1}^n]) > 0$

$$u_n(t) = u_i^n + \frac{\mu(]t_i^n, t])}{\mu(]t_i^n, t_{i+1}^n])}(u_{i+1}^n - u_i^n) \quad \text{for all } t \in [t_i^n, t_{i+1}^n[.$$

According to (4.10) we can write

$$u_n(t) = u_i^n = u_{i+1}^n \quad \text{for all } t \in [t_i^n, t_{i+1}^n], \text{ if } \mu(]t_i^n, t_{i+1}^n]) = 0, \tag{4.11}$$

and by what precedes we can also write

$$u_n(t) = u_i^n + \frac{\mu(]t_i^n, t])}{\mu(]t_i^n, t_{i+1}^n])}(u_{i+1}^n - u_i^n) \quad \text{for all } t \in [t_i^n, t_{i+1}^n], \text{ if } \mu(]t_i^n, t_{i+1}^n]) > 0. \tag{4.12}$$

We observe on the one hand that $u_n(\cdot)$ is right continuous on I and $u_n(t_i^n) = U_n(t_i^n)$ for all $i = 0, \cdots, p(n)$, and on the other hand, thanks to (4.6) again,

$$\|u_n(t)\| \le \rho_0 \quad \text{for all } t \in I. \tag{4.13}$$

Further, the definitions of $u_n(\cdot)$ and $U_n(\cdot)$ combined with (4.7) give, for every $t \in]t_i^n, t_{i+1}^n[$, that $\|u_n(t) - U_n(t)\| \le \varepsilon_n$. Therefore,

$$\|u_n(t) - U_n(t)\| \le \varepsilon_n \quad \text{for all } t \in I,$$

so by Step 1 the sequence $(u_n(\cdot))_n$ converges uniformly on I to the mapping $u(\cdot)$ obtained in Step 1.

On the other hand, the definition of $u_n(\cdot)$, through (4.11) and (4.12), can be rewritten, for any $t \in I$, as

$$u_n(t) = u_n(T_0) + \int_{]T_0, t]} \zeta_n(s)\, d\mu(s),$$

where $\zeta_n(T_0) = 0$ and if $\mu(]t_i^n, t_{i+1}^n]) = 0$ we have $\zeta_n(t) = 0$ for all $t \in]t_i^n, t_{i+1}^n]$, and if $\mu(]t_i^n, t_{i+1}^n]) > 0$

$$\zeta_n(t) := \frac{u_{i+1}^n - u_i^n}{\mu(]t_i^n, t_{i+1}^n])} \quad \text{for all } t \in]t_i^n, t_{i+1}^n]. \tag{4.14}$$

Consequently, each mapping $u_n(\cdot)$ is right continuous with bounded variation on the whole interval I. The latter equality tells us also that the vector measure du_n has the latter integrand $\zeta_n(\cdot)$ as a density relative to μ, so by the first equality in (2.11)

$$\frac{du_n}{d\mu}(\cdot) \text{ is a density of } du_n \text{ relative to } \mu,$$

and, for μ-almost every $t \in I$,

$$\frac{du_n}{d\mu}(t) = \zeta_n(t). \tag{4.15}$$

Taking the definition of $\zeta_n(t)$ and (4.7) into account, it results that

$$\left\| \frac{du_n}{d\mu}(t) \right\| = \|\zeta_n(t)\| \le 1 \quad \mu - \text{a.e. } t \in I. \tag{4.16}$$

According to this inequality, extracting a subsequence if necessary we may and do suppose that the sequence $\big(\zeta_n = \frac{du_n}{d\mu}\big)_n$ converges weakly in $L^2_\mu(I, H)$ to some mapping $\zeta(\cdot)$. From this and the equality

$$u_n(t) = u_0 + \int_{]T_0,t]} \zeta_n(s)\, d\mu(s),$$

we get for every $t \in I$

$$u(t) = u_0 + \int_{]T_0,t]} \zeta(s)\, d\mu(s).$$

This says in particular that the mapping $u(\cdot)$ is right continuous with bounded variation and that it admits the mapping $\zeta(\cdot)$ as a density relative to μ, so $\frac{du}{d\mu}(\cdot) = \zeta(\cdot)$ μ-almost everywhere and

$$u(t) = u_0 + \int_{T_0,t]} \frac{du}{d\mu}(s)\, d\mu(s) \quad \text{for all } t \in I.$$

Note that (4.16) also entails that $\|\frac{du}{d\mu}(t)\| \le 1$ for μ-almost all $t \in I$, hence

$$\|u(t) - u(s)\| \le \mu(]s,t]), \quad \text{for all } s, t \in I \text{ with } s \le t,$$

which is the desired inequality property of the theorem. Further, this inequality clearly yields for every $t \in I$ that

$$\|u(t)\| \le \|u_0\| + \mu(]T_0,T]), \text{ so } \|u(t)\| \le \min\{\rho_0, \|u_0\| + \mu(]T_0,T])\}.$$

Now, for each $n \in \mathbb{N}$ define the function $\theta_n(\cdot) : I \to I$ by $\theta_n(T_0) := T_0$ and

$$\theta_n(t) := t^n_{i+1} \quad \text{if } t \in]t^n_i, t^n_{i+1}] \ (0 \le i \le p(n) - 1), \tag{4.17}$$

and note by (2.1), (4.14) and (4.15) that, for μ-almost every $t \in I$,

$$\zeta_n(t) = \frac{du_n}{d\mu}(t) \in -N\big(C(\theta_n(t)); u_n(\theta_n(t))\big).$$

This yields according to (2.4) and (4.16), for μ-almost every $t \in I$

$$\frac{du_n}{d\mu}(t) \in -\partial d_{C(\theta_n(t))}\big(u_n(\theta_n(t))\big), \tag{4.18}$$

so we can follow the idea (introduced in our paper [72]) of reduction of some aspects of sweeping process to a suitable differential inclusion associated with the subdifferential of the distance function to the moving set.

Define also the function $\delta_n(\cdot) : I \to I$ by $\delta_n(T) = T$ and $\delta_n(t) = t_i^n$ if $t \in [t_i^n, t_{i+1}^n[$.

Step 3. Let us prove that $u(\cdot)$ is a solution.
We have already established above that the differential measure du is absolutely continuous with respect to μ.

Let us show that $u(t) \in C(t)$ for all $t \in I$. First, for each $t \in I$, writing by (4.16)

$$\|u_n(\theta_n(t)) - u(t)\| \le \|u_n(t) - u(t)\| + \mu(]t, \theta_n(t)]),$$

we see that, as $n \to \infty$,

$$u_n(\theta_n(t)) \to u(t). \tag{4.19}$$

Similarly, writing

$$\|u_n(\delta_n(t)) - u(t)\| \le \|u_n(t) - u(t)\| + \mu(]\delta_n(t), t]),$$

we get for any point t where $\mu(\{t\}) = 0$ that

$$u_n(\delta_n(t)) \to u(t) \quad \text{as } n \to \infty. \tag{4.20}$$

Fix any $t \in]T_0, T[$. If $\mu(\{t\}) = 0$, we note from the retraction of $C(\cdot)$ along ρ-truncation, from (2.6), and from the inclusion $u_n(\delta_n(t)) \in C(\delta_n(t)) \cap \rho\mathbb{B}$ (see (4.13)), that

$$d_{C(t)}\big(u_n(\delta_n(t))\big) \le d\big(u_n(\delta_n(t)), C(\delta_n(t)) \cap \rho\mathbb{B}\big) + \mu(]\delta_n(t), t]) = \mu(]\delta_n(t), t]).$$

If $\mu(\{t\}) = 0$, it then results that $u(t) \in C(t)$ according to (4.20) and to the closedness of $C(t)$.

Suppose now $\mu(\{t\}) > 0$. Here, the above construction of the sequence of mappings $(U_n(\cdot))_n$ will be involved in a fundamental way. For each $n \in \mathbb{N}$ there is some $i_n \in \{0, \cdots, p(n) - 1\}$ such that $t \in [t_{i_n}^n, t_{i_n+1}^n[$. Choose $k \in \mathbb{N}$ such that $\varepsilon_n < \mu(\{t\})$ for all $n \ge k$. This and the inequality $\mu(]t_{i_n}^n, t_{i_n+1}^n[) < \varepsilon_n$ in (4.4) entails for every $n \ge k$ that $t \notin]t_{i_n}^n, t_{i_n+1}^n[$. Consequently, for every $n \ge k$ we have $t = t_{i_n}^n$, so

$$U_n(t) = U_n(t_{i_n}^n) \in C(t_{i_n}^n) = C(t).$$

Combining this with the convergence $U_n(t) \underset{n\to\infty}{\longrightarrow} u(t)$ and with the closedness of $C(t)$, we obtain $u(t) \in C(t)$.

Since $U_n(T_0) \in C(T_0)$ and $U_n(T) \in C(T)$ by the construction of U_n, we also have $u(T_0) = u_0 \in C(T_0)$ and $u(T) \in C(T)$, so $u(t) \in C(t)$ for all $t \in I$.

It remains to show that

$$\frac{du}{d\mu}(t) \in -N\big(C(t); u(t)\big) \;\; \mu - \text{a.e. } t \in I,$$

which will be obtained in a standard way.

Since the sequence $(\zeta_n(\cdot))_n$ converges weakly in $L^2_\mu(I, H)$ to $\zeta(\cdot)$, by Mazur's lemma there exists a sequence $(\xi_n(\cdot))_n$ converging strongly in $L^2_\mu(I, H)$ to $\zeta(\cdot)$ with

$$\xi_n(\cdot) \in \operatorname{conv}\{\zeta_k : \; k \ge n\}, \quad \text{for all } n \in \mathbb{N}.$$

This sequence $(\xi_n(\cdot))_n$ has a subsequence (that we do not relabel) converging μ-almost everywhere to $\zeta(\cdot)$, hence, there is some Borel set $I_0 \subset I$ with $\mu(I \setminus I_0) = 0$ such that, for all $t \in I_0$,

$$\zeta(t) \in \bigcap_n \overline{\mathrm{conv}}\{\zeta_k(t) : \; k \geq n\},$$

where $\overline{\mathrm{conv}}$ denotes the closed convex hull in H. Fixing any $t \in I_0$ and any $h \in H$, it results from (4.18) that

$$\langle -\zeta(t), h\rangle \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} \langle -\zeta_k(t), h\rangle \leq \limsup_{n\to\infty} d'_{C(\theta_n(t))}\big(u_n(\theta_n(t)); h\big),$$

and, since $\theta_n(t) \geq t$ with $\mu(]t, \theta_n(t)]) \to 0$ and $u_n(\theta_n(t)) \to u(t)$ (see (4.19)) with $u_n(\theta_n(t)) \in C\big(\theta_n(t)\big)$ and $u(t) \in C(t)$ and since $\|u(t)\| \leq \rho_0 < \rho$ (see (4.13)), Proposition 4.1 implies that

$$\langle -\zeta(t), h\rangle \leq d'_{C(t)}\big(u(t); h\big).$$

The latter inequality means, for each $t \in I_0$, that $-\zeta(t) \in \partial d_{C(t)}(u(t))$. Since $u(t) \in C(t)$, the first equality in (2.4) yields

$$\frac{du}{d\mu}(t) = \zeta(t) \in -N\big(C(t); u(t)\big),$$

which finishes the proof of the theorem. ■

The solution mapping $u : I \to H$ in the above theorem being with bounded variation we know that $u(t^-) := \lim_{s\uparrow t} u(s)$ exists for all $t \in]T_0, T]$, where we recall that $s \uparrow t$ means $s \to t$ with $s < t$. The next corollary provides the link between $u(t^-)$ and the set $C(t)$.

Corollary 4.1 *Under the assumptions of Theorem 4.1, the solution $u(\cdot)$ of the measure differential inclusion of the theorem enjoys the properties*

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \textit{and} \quad u(t) = \mathrm{proj}_{\,C(t)}\big(u(t^-)\big) \quad \textit{for all}\, t \in]T_0, T].$$

Proof. Fix any $t \in]T_0, T]$. We know by Theorem 4.1 that $\|u(t) - u(s)\| \leq \mu(]s, t])$ for all $t \in]T_0, T[$, so $\|u(t) - u(t^-)\| \leq \mu(\{t\})$. This says in particular that, if $\mu(\{t\}) = 0$ we have $u(t^-) = u(t)$, hence $u(t) = \mathrm{proj}_{\,C(t)}\big(u(t^-)\big)$ since $u(t) \in C(t)$ according to the definition of a solution. Suppose now that $\mu(\{t\}) > 0$. It results from (2.12) that

$$\frac{du}{d\mu}(t) = \lim_{r\downarrow 0} \frac{du(]t - r, t] \cap I)}{d\mu(]t - r, t] \cap I)} = \frac{u(t) - u(t^-)}{\mu(\{t\})}.$$

Since $\mu(\{t\}) > 0$, Definition 2.1 again entails that at this point t we must have

$$\frac{du}{d\mu}(t) \in -N\big(C(t); u(t)\big), \quad \text{hence } \frac{u(t) - u(t^-)}{\mu(\{t\})} \in -N\big(C(t); u(t)\big),$$

so according to the cone property of $N\big(C(t); u(t)\big)$ we deduce that

$$u(t^-) - u(t) \in N\big(C(t); u(t)\big).$$

This and (2.2) guarantee that $u(t) = \operatorname{proj}_{C(t)}\big(u(t^-)\big)$, which finishes the proof. ■

Under the convexity of the sets $C(t)$ we know by Corollary 3.2 that there is no privilege of measures absolutely continuously equivalent to the measure associated with the ρ-retraction.

In addition to that property and to Corollary 4.1 let us establish through Proposition 3.1 another corollary relative to the absolute continuity setting.

Corollary 4.2 *Let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed convex sets of the Hilbert space H and let $u_0 \in C(T_0)$. Assume that there are a non-decreasing absolutely continuous function $v : I \to \mathbb{R}_+$, a real $\rho_0 \geq \|u_0\|$ and an extended real $\rho > \rho_0$ such that*

$$\operatorname{exc}_\rho\big(C(s), C(t)\big) \leq v(t) - v(s) \quad \textit{for all } s \leq t \textit{ in } I,$$

and such that, for all $t_1 < \cdots < t_k$ in I the inequality

$$\|\big(\operatorname{proj}_{C(t_k)} \circ \cdots \circ \operatorname{proj}_{C(t_1)}\big)(u_0)\| \leq \rho_0$$

is satisfied.

Then the differential inclusion

$$\begin{cases} \frac{du}{dt}(t) \in -N\big(C(t); u(t)\big) \\ u(T_0) = u_0, \end{cases}$$

has one and only one usual absolutely continuous solution.

Furthermore, the solution $u(\cdot)$ satisfies the inequality

$$\|u(t) - u(s)\| \leq v(t) - v(s) \quad \textit{for all } s \leq t \textit{ in } I,$$

and one has for all $t \in I$

$$\|u(t)\| \leq \min\{\rho_0, \|u_0\| + v(T) - v(T_0)\}.$$

Proof. Consider the positive Radon measure μ on $I = [T_0, T]$ defined by

$$\mu(]s,t]) = v(t) - v(s), \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

With this measure μ at hand, Theorem 4.1 tells us that the measure differential inclusion (2.7) has one and only one solution $u(\cdot)$ in the sense of Definition 2.1, and this solution satisfies the inequality $\|u(t) - u(s)\| \leq \mu(]s,t])$ for all $s \leq t$ in I. Noting that μ is absolutely continuous with respect to the Lebesgue measure according to the absolute continuity of $v(\cdot)$, Proposition 3.1 guarantees that $u(\cdot)$ is the unique usual absolutely continuous solution of the differential inclusion of the theorem. ■

From Theorem 4.1 and Corollary 4.1, we can derive one of the results of Moreau [56] (see Proposition 3.b in [56]).

Corollary 4.3 (Moreau [56]) *Let $C : I \rightrightarrows H$ be a set-valued mapping from $I := [T_0, T]$ into the nonempty closed convex subsets of the Hilbert space H, and let $u_0 \in C(T_0)$. Assume that there exists a positive Radon measure μ on I such that, for all $s, t \in I$ with $s \leq t$*

$$\mathrm{exc}\big(C(s), C(t)\big) \leq \mu(]s, t]).$$

Then, the following sweeping process

$$\begin{cases} -du \in N\big(C(t); u(t)\big) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right-continuous with bounded variation solution.

Furthermore, the solution $u(\cdot)$ satisfies the inequality

$$\|u(t) - u(s)\| \leq \mu(]s, t]) \quad \textit{for all } s, t \in I \textit{ with } s \leq t.$$

In particular, for all $t \in I$ one has

$$\|u(t) - u(t^-)\| \leq \mu(\{t\}) \quad \textit{and} \quad \|u(t)\| \leq \|u_0\| + \mu(]T_0, T]).$$

Proof. Let $\rho_0 := \|u_0\| + \mu(]T_0, T])$. Consider any $t_1 < \cdots < t_k$ in I and put $r_i := \mu(]t_{i-1}, t_i])$ for $i = 1, \cdots, k$, where $t_0 := T_0$. Put also

$$u_i := \mathrm{proj}_{\,C(t_1)} \circ \cdots \circ \mathrm{proj}_{\,C(t_i)}(u_0) \quad \text{for } i = 1, \cdots, k.$$

From the inclusion $u_0 \in C(t_0)$ and the inequality $\mathrm{exc}\big(C(t_0), C(t_1)\big) \leq r_1$ we see that

$$\begin{aligned} \|u_1\| &\leq \|u_0\| + \|u_0 - u_1\| = \|u_0\| + d_{C(t_1)}(u_0) \\ &\leq \|u_0\| + \mathrm{exc}\big(C(t_0), C(t_1)\big) \leq \|u_0\| + r_1. \end{aligned}$$

Similarly, from the inclusion $u_1 \in C(t_1)$ and the inequality $\mathrm{exc}\big(C(t_1), C(t_2)\big) \leq r_2$ we obtain $\|u_2\| \leq \|u_1\| + r_2 \leq \|u_0\| + r_1 + r_2$. By induction it follows that

$$\|u_k\| \leq \|u_0\| + r_1 + \cdots + r_k = \|u_0\| + \mu(]t_0, t_k]) \leq \rho_0.$$

Consequently, the positive Radon measure μ, the above real ρ_0 and the extended real $\rho := +\infty$ fulfill the conditions in Theorem 4.1. The corollary then follows from that theorem and Corollary 4.1. ■

Various examples fulfilling the conditions of the above corollary have been given by J.J. Moreau, e.g., in [53, 54]. Notice in passing that, given any closed convex set S and any right-continuous with bounded variation mapping $\zeta : [T_0, T] \to H$, the sets $C(t) := \zeta(t) + S$ satisfy the conditions. Indeed, for any $s, t \in [T_0, T]$ with $s \leq t$ it is easily seen that

$$\mathrm{var}\big(C(s), C(t)\big) \leq \|\zeta(s) - \zeta(t)\| = \|(d\zeta)(]s, t])\| \leq |d\zeta|(]s, t]),$$

where we recall that $|d\zeta|$ denotes the variation measure of the differential measure $d\zeta$.

We proceed now to provide some existence results under some conditions on the mapping $t \mapsto \text{proj}_{C(t)}(a)$ (with some $a \in H$) ensuring the bounded retraction along ρ-truncation of $C(\cdot)$. For this, we will use *some main ideas from* [25]; see the statement of Theorem 2.1 therein.

Theorem 4.2 *Let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed convex sets of the Hilbert space H for which there exists some $a \in H$ such that the mapping $t \mapsto \text{proj}_{C(t)}(a)$ is of bounded variation, that is,*

$$W := \text{var}\big(\text{proj}_{C(\cdot)}(a); I\big) = \sup\big(\sum_{i=1}^{n} \|\text{proj}_{C(t_{i+1})}(a) - \text{proj}_{C(t_i)}(a)\|\big) < +\infty,$$

where in the supremum $n \in \mathbb{N}$ and $t_1 < \cdots < t_n$ in I. Let also $u_0 \in C(T_0)$. Assume that, for the real number

$$\rho_0 := \|u_0 - a\| + W + \sup_{t \in I} \|\text{proj}_{C(t)}(a)\|,$$

there exists a positive Radon measure μ on I and an extended real $\rho > \rho_0$ such that, for all $s, t \in I$ with $s \le t$

$$\text{exc}_\rho\big(C(s), C(t)\big) \le \mu(]s,t]).$$

Then the sweeping process

$$\begin{cases} -du \in N\big(C(t); u(t)\big) \\ u(T_0) = u_0 \end{cases}$$

has one and only one right-continuous with bounded variation solution.

Furthermore, the solution $u(\cdot)$ satisfies the inequality

$$\|u(t) - u(s)\| \le \mu(]s,t]) \quad \textit{for all } s \le t \textit{ in } I,$$

and for all $t \in I$

$$\|u(t) - u(t^-)\| \le \mu(\{t\}) \;\; \textit{and} \;\; \|u(t)\| \le \min\{\rho_0, \|u_0\| + \mu(]T_0, T])\}.$$

Proof. Let $t_1 < \cdots < t_k$ in I and put

$$u_1 := \text{proj}_{C(t_1)}(u_0), \quad \text{and } u_i := \text{proj}_{C(t_i)}(u_{i-1}),$$

for $i = 1, \cdots, k$. For each $i \in \{0, 1, \cdots, k\}$ put

$$y_{i,i} := \text{proj}_{C(t_i)}(a), \; y_{i,i-1} := \big(\text{proj}_{C(t_i)} \circ \text{proj}_{C(t_{i-1})}\big)(a), \cdots,$$
$$y_{i,0} := \big(\text{proj}_{C(t_i)} \circ \cdots \circ \text{proj}_{C(t_0)}\big)(a),$$

so fixing any $i \in \{1, \cdots, k\}$ we see that, for every $j \in \{0, \cdots, i-1\}$

$$
\begin{aligned}
&\|y_{i,j+1} - y_{i,j}\| \\
&\le \|\operatorname{proj}_{C(t_i)} \circ \cdots \circ \operatorname{proj}_{C(t_{j+1})}(a) - \operatorname{proj}_{C(t_i)} \circ \cdots \circ \operatorname{proj}_{C(t_{j+1})} \circ \operatorname{proj}_{C(t_j)}(a)\| \\
&\le \|\operatorname{proj}_{C(t_{j+1})}(a) - \operatorname{proj}_{C(t_{j+1})} \circ \operatorname{proj}_{C(t_j)}(a)\| \\
&= \|\operatorname{proj}_{C(t_{j+1})} \circ \operatorname{proj}_{C(t_{j+1})}(a) - \operatorname{proj}_{C(t_{j+1})} \circ \operatorname{proj}_{C(t_j)}(a)\|,
\end{aligned}
$$

hence

$$\|y_{i,j+1} - y_{i,j}\| \le \|\operatorname{proj}_{C(t_{j+1})}(a) - \operatorname{proj}_{C(t_j)}(a)\|.$$

Thus, it ensues that

$$
\begin{aligned}
\|u_i - y_{i,i}\| &\le \Big(\sum_{j=0}^{i-1} \|y_{i,j+1} - y_{i,j}\|\Big) + \|y_{i,0} - u_i\| \\
&\le \Big(\sum_{j=0}^{i-1} \|\operatorname{proj}_{C(t_{j+1})}(a) - \operatorname{proj}_{C(t_j)}(a)\|\Big) + \|y_{i,0} - u_i\| \\
&\le W + \|y_{i,0} - u_i\|.
\end{aligned}
$$

On the other hand, since $u_0 = \operatorname{proj}_{C(t_0)}(u_0)$, we also have

$$
\begin{aligned}
\|y_{i,0} - u_i\| &= \|\operatorname{proj}_{C(t_i)} \circ \cdots \circ \operatorname{proj}_{C(t_0)}(a) - \operatorname{proj}_{C(t_i)} \circ \cdots \circ \operatorname{proj}_{C(t_0)}(u_0)\| \\
&\le \|a - u_0\|.
\end{aligned}
$$

Consequently, we obtain

$$\|u_i - y_{i,i}\| \le W + \|u_0 - a\|.$$

Observing that

$$\|y_{i,i}\| = \|\operatorname{proj}_{C(t_i)}(a)\| \le \sup_{t \in I} \|\operatorname{proj}_{C(t)}(a)\| =: \gamma,$$

it results that

$$\|u_i\| \le \gamma + W + \|u_0 - a\| \quad \text{for all } i \in \{1, \cdots, k\}.$$

We can then apply Theorem 4.1 to derive the corollary. ■

From the above theorem and from Proposition 3.1 we directly derive the following corollary which is in the line of Theorem 2.1 in [25] and Theorem 3.1 [74].

Corollary 4.4 *Let $C : I \rightrightarrows H$ be a set-valued mapping with nonempty closed convex sets of the Hilbert space H for which there exists some $a \in H$ such that the mapping $t \mapsto \operatorname{proj}_{C(t)}(a)$ is of bounded variation, and let*

$$W := \operatorname{var}\big(\operatorname{proj}_{C(\cdot)}(a); I\big) < +\infty.$$

Let also $u_0 \in C(T_0)$. *Assume that, for the real number*

$$\rho_0 := \|u_0 - a\| + W + \sup_{t\in I} \|\mathrm{proj}_{\,C(t)}(a)\|,$$

there exists a non-decreasing absolutely continuous function v *on* I *and an extended real* $\rho > \rho_0$ *such that*

$$\mathrm{exc}_\rho\big(C(s), C(t)\big) \le v(t) - v(s) \quad \textit{for all } s,t \in I \textit{ with } s \le t.$$

Then the sweeping process

$$\begin{cases} -\frac{du}{dt} \in N\big(C(t); u(t)\big) \\ u(T_0) = u_0 \end{cases}$$

has one and only one usual absolutely continuous solution on I.

Furthermore, the solution $u(\cdot)$ *satisfies the inequality*

$$\|u(t) - u(s)\| \le v(t) - v(s) \quad \textit{for all } s \le t \textit{ in } I,$$

and for all $t \in I$

$$\|u(t)\| \le \min\{\rho_0, \|u_0\| + v(T) - v(T_0)\}.$$

Clearly the assumption of bounded variation concerning the mapping $t \mapsto \mathrm{proj}_{\,C(t)}(a)$ (for some $a \in H$) is fulfilled whenever there is a common point a in all the sets $C(t)$.

Another example is furnished by the following proposition *strongly inspired from* Theorem 4.1 and Lemma 4.2 in [74] and from Corollary 2.2 and Proposition 2.3 in [25]. The example involves a mapping $\zeta : I \to H$ with bounded variation on $I := [T_0, T]$ as well as the positive variation measure $|d\zeta|$ of its associated differential measure $d\zeta$.

Proposition 4.2 *Let* $\zeta : I \to H$ *and* $\beta : I \to \mathbb{R}$ *be two right-continuous with bounded variation mappings on* $I := [T_0, T]$ *with* $\|\zeta(t)\| = 1$ *for all* $t \in I$, *and let*

$$C_0(t) := \{x \in H : \langle \zeta(t), x\rangle = \beta(t)\} \quad C_1(t) := \{x \in H : \langle \zeta(t), x\rangle \le \beta(t)\}.$$

Then, for each $i = 0, 1$ *the mapping* $t \mapsto \mathrm{proj}_{\,C_i(t)}(0)$ *is of bounded variation on* I *and, for any real* $\rho > 0$ *one has, for all* s, t *in* I *with* $s < t$

$$\mathrm{exc}_\rho\big(C_i(s), C_i(t)\big) \le \rho \int_{]s,t]} |d\zeta| + \int_{]s,t]} |d\beta| = \big(\rho|d\zeta| + |d\beta|\big)(]s,t]).$$

In particular, for each $i = 0, 1$, *given* $u_{i,0} \in C_i(T_0)$ *the sweeping process differential inclusion*

$$\begin{cases} -du \in N\big(C_i(t); u(t)\big) \\ u(T_0) = u_{i,0} \end{cases}$$

has one and only one right-continuous with bounded variation solution.

Proof. Let us start with $i=0$. For any $t \in I$ it is known that

$$d\big(x, C_0(t)\big) = \frac{|\langle \zeta(t), x\rangle - \beta(t)|}{\|\zeta(t)\|} = |\langle \zeta(t), x\rangle - \beta(t)|,$$

so $d\big(0, C(t)\big) = |\beta(t)|$. Since $\beta(t)\zeta(t) \in C_0(t)$, it follows that $\mathrm{proj}_{C_0(t)}(0) = \beta(t)\zeta(t)$, hence the mapping $t \mapsto \mathrm{proj}_{C_0(t)}(0)$ is of bounded variation on the bounded interval $I = [T_0, T]$.

Given any real $\rho > 0$ and any $s, t \in I$ with $s < t$, from the first equality above we also have, for every $x \in C_0(s)$

$$\begin{aligned} d\big(x, C_0(t)\big) &= |\langle \zeta(t), x\rangle - \beta(t)| \\ &= |\langle \zeta(t), x\rangle - \beta(t) - \big(\langle \zeta(s), x\rangle - \beta(s)\big)| \\ &\le \|x\|\,\|\zeta(t) - \zeta(s)\| + |\beta(t) - \beta(s)|, \end{aligned}$$

which entails for every $x \in C_0(s) \cap \rho\mathbb{B}$ that

$$\begin{aligned} d\big(x, C_0(t)\big) &\le \rho\|\zeta(t) - \zeta(s)\| + |\beta(t) - \beta(s)| = \rho\left\|\int_{]s,t]} d\zeta\right\| + \left|\int_{]s,t]} d\beta\right| \\ &\le \rho\int_{]s,t]} |d\zeta| + \int_{]s,t]} |d\beta| = \big(\rho|d\zeta| + |d\beta|\big)(]s,t]). \end{aligned}$$

Therefore, we see that, for any $s, t \in I$ with $s < t$

$$\mathrm{exc}_\rho\big(C_0(s), C_0(t)\big) \le \big(\rho|d\zeta| + |d\beta|\big)(]s,t]).$$

Concerning the other case $i=1$, observe first that, for $0 \in C_1(t)$ we have $d\big(0, C_1(t)\big) = 0$, and for $0 \notin C_1(t)$

$$d\big(0, C_1(t)\big) = \frac{|\langle \zeta(t), 0\rangle - \beta(t)|}{\|\zeta(t)\|} = |\beta(t)|,$$

so in any case $d\big(0, C_1(t)\big) = \big(-\beta(t)\big)^+$, where $r^+ := \max\{r, 0\}$. This and the inclusion $-\big(-\beta(t)\big)^+\zeta(t) \in C_1(t)$ yield that $\mathrm{proj}_{C_1(t)}(0) = -\big(-\beta(t)\big)^+\zeta(t)$. In particular, this says that the mapping $t \mapsto \mathrm{proj}_{C_1(t)}(0)$ is of bounded variation on $I = [T_0, T]$, since the function $t \mapsto \big(-\beta(t)\big)^+$ is of bounded variation according to the Lipschitzian property over $\mathbb{R}$ of the function $r \mapsto r^+$.

On the other hand, note that, for any $s, t \in I$ and any $x \in C_1(s)$ with $x \notin C_1(t)$

$$\langle \zeta(s), x\rangle - \beta(s) \le 0 \quad \text{and} \quad \langle \zeta(t), x\rangle - \beta(t) > 0,$$

which gives

$$d\big(x, C_1(t)\big) = \frac{|\langle \zeta(t), x\rangle - \beta(t)|}{\|\zeta(t)\|} = \langle \zeta(t), x\rangle - \beta(t). \tag{4.21}$$

Consider now any real $\rho > 0$ and any $s < t$ in I. For any $x \in C_1(s) \cap \rho\mathbb{B}$ with $x \notin C_1(t)$, it results from the latter equality that

$$\begin{aligned} d\big(x, C_1(t)\big) &= \langle \zeta(t) - \zeta(s), x\rangle - \big(\beta(t) - \beta(s)\big) + \langle \zeta(s), x\rangle - \beta(s) \\ &\le \langle \zeta(t) - \zeta(s), x\rangle - \big(\beta(t) - \beta(s)\big), \end{aligned}$$

hence we see as above that

$$\begin{aligned} d\big(x, C_1(t)\big) &\le \rho\|\zeta(t) - \zeta(s)\| + |\beta(t) - \beta(s)| \\ &\le \rho \int_{]s,t]} |d\zeta| + \int_{]s,t]} |d\beta| = \big(\rho|d\zeta| + |d\beta|\big)(]s,t]). \end{aligned}$$

This and the equality $d\big(x, C_1(t)\big) = 0$ when $x \in C_1(t)$ justify that, for $s < t$ in I

$$\mathrm{exc}_\rho\big(C_1(s), C_1(t)\big) \le \big(\rho|d\zeta| + |d\beta|\big)(]s,t]).$$

Consequently, putting

$$W_i := \mathrm{var}\big(\mathrm{proj}_{C_i(\cdot)}(0); I\big) \quad \text{and} \quad \rho_{i,0} := \|u_0\| + W_i + \sup_{t\in I} \|\mathrm{proj}_{C_i(t)}(0)\|,$$

and taking any real $\rho > \rho_{i,0}$, the assumptions of Theorem 4.1 are satisfied. The conclusion of the same theorem then finishes the proof. ■

First, we must emphasize that, in general, for such sets $C(t)$ like in the statement of Proposition 4.2 there exists no positive Radon measure μ on $I = [T_0, T]$ satisfying

$$\mathrm{exc}\big(C(s), C(t)\big) \le \mu(]s,t]) \quad \text{for all } s < t \text{ in } I.$$

Take for example $C(t)$ as $C_1(t)$ in Proposition 4.2. If for some $s, t \in I$ with $t > s$ we have $\zeta(t) \notin \mathbb{R}_+\zeta(s)$, then there is some $\bar{x} \in H$ with $\langle \zeta(s), \bar{x}\rangle \le 0$ and $\langle \zeta(t), \bar{x}\rangle > 0$. Fixing a point $z \in C(s)$ and setting $x_n := z + n\bar{x}$, it is clear that $x_n \in C(s)$ for all n and that $\langle \zeta(t), x_n\rangle \to +\infty$ as $n \to \infty$. Therefore, we obtain $x_n \notin C(t)$ for large n and by (4.21)

$$d\big(x_n, C(t)\big) \to +\infty \text{ as } n \to \infty, \ \text{ thus } \mathrm{exc}\big(C(s), C(t)\big) = +\infty.$$

Recalling that the measure of any compact set by a Radon measure is finite, the latter equality confirms that there is no positive Radon measure μ such that $\mathrm{exc}\big(C(s), C(t)\big) \le \mu(]s,t])$ for all $s, t \in I$ with $s < t$.

It is also worth pointing out that the proof of Proposition 4.2 makes clear that, if the mappings $\zeta(\cdot)$ and $\beta(\cdot)$ are absolutely continuous (resp. Lipschitzian) on I, then (for each $i = 0, 1$) the solution of the sweeping process in that proposition is a usual absolutely continuous (resp. Lipschitzian) solution.

Conclusion. After establishing various general basic properties of solutions of convex as well as nonconvex sweeping processes with *bounded truncated retraction*, we proved in Theorem 4.1 and Theorem 4.2 the existence and uniqueness of solutions under general verifiable assumptions for convex sweeping processes with bounded truncated retraction. Proposition 4.2 illustrates (as in [25, 74]) the case of *unbounded* polyhedral convex sets $C(t)$, and the comments following that proposition show that for such sets the control of the retraction in the form

$$\operatorname{exc}\big(C(s), C(t)\big) \leq \mu(]s,t]) \quad \text{for } s < t$$

with a positive Radon measure μ fails in general. This confirms the interest and motivation of the analysis of convex sweeping processes whose some suitable ρ-truncated retraction $t \mapsto \operatorname{ret}_\rho(C;T_0,t)$ (and not the usual retraction $t \mapsto \operatorname{ret}(C;T_0,t)$) is bounded. The study of existence results in the framework of sweeping processes with prox-regular sets $C(t)$ such that $t \mapsto \operatorname{ret}_\rho(C;T_0,t)$ is bounded will be the subject of another paper.

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