Subdifferential characterization of s-lower regular functions

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Abstract

As for Moreau envelopes of primal lower nice as well as prox-regular functions, Moreau s-envelopes of s-lower regular functions have been proved in our paper [17] to have several remarkable differential properties and to have many important applications. Here we provide a subdifferential characterization of extended real-valued s-lower regular functions on Banach spaces in terms of a hypomonotonicity-like property of the subdifferential.

Keywords. Subdifferential, lower regular function, convexly composite function, primal lower nice function, hypomonotonicity

1 Introduction

In his 1990 paper [28] R.A. Poliquin showed that an extended real-valued proper lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if its proximal subdifferential $\partial_P f$ is monotone in the sense that for all (x_i, ζ_i) in $\mathbb{R}^n \times \mathbb{R}^n$ with $\zeta_i \in \partial_P f(x_i)$, $i = 1, 2$, the monotonicity inequality

$$\langle \zeta_1 - \zeta_2, x_1 - x_2 \rangle \geq 0$$

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is satisfied; for the extension of that subdifferential characterization to Banach space, we refer to R. Correa, A. Jofre and L. Thibault [13, 14]. One year later, Poliquin [29] introduced the class of primal lower nice functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and proved that an extended real-valued function f on $\mathbb{R}^n$ is primal lower nice if and only if its proximal subdifferential is hypomonotone. Poliquin also showed how the theory of second order analysis for convex functions can be extended to the class of nonconvex primal lower nice functions.

In view of the second order analysis for a larger class of nonconvex functions on $\mathbb{R}^n$, R.A. Poliquin and R.T. Rockafellar [30] introduced the class of prox-regular functions from $\mathbb{R}^n$ to $\mathbb{R}\cup\{+\infty\}$. In addition to the study of second order analysis, Poliquin and Rockafellar [30] also established a complete characterization of prox-regular functions on $\mathbb{R}^n$ (see also [32]). Those functions as well as the primal lower nice ones arise in the study of prox-regular sets; we refer to [31, 4, 5, 10]. Properties of primal lower nice functions on Hilbert spaces are studied by L. Thibault and D. Zagrodny [34], A.B. Levy, R.A. Poliquin and L. Thibault [23], M. Ivanov and N. Zlateva [19, 20], F. Bernard, L. Thibault and D. Zagrodny [3], O.S. Serea and L. Thibault [33], M. Mazade and L. Thibault [25]; we also refer to [15] for the larger class of Φ-convex functions and to [24] for applications to evolution problems. Properties of prox-regular functions on Hilbert spaces are analyzed in details in [2, 6] where among others the finite-dimensional characterization is generalized to Hilbert spaces; see also [12, 1, 22]. Various other subregularities can be found in [8, 21, 22].

For a lower semicontinuous function f from a Hilbert space X into $\mathbb{R} \cup \{+\infty\}$, its primal lower nice property over an open convex set $\mathcal{O}$ of H can be seen as the inequality

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - c(1 + \|\zeta\|)\|x - y\|^2$$

for all $x, y \in \mathcal{O}$ and ζ in the Clarke subdifferential of f at x (see the next section for the definition). Assuming that X is a Banach space and $s > 0$, the s-lower regularity corresponds to the extension

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - c(1 + \|x^*\|)\|x - y\|^{s+1}$$

for all $x, y \in \mathcal{O}$ and x^* in the Clarke subdifferential of f at x. The Moreau s-envelopes of such functions are investigated in our work [17]. The aim of the present paper is to provide a complete subdifferential characterization of the

s-lower regularity in the Banach setting, generalizing in this way the results recalled above for primal lower nice functions in $\mathbb{R}^n$ or in Hilbert spaces.

Section 2 recalls some concepts and results used throughout the paper. The theorem proving the subdifferential characterization of extended real-valued s-lower regular functions on Banach spaces is established in Section 3. In the same section we also show that for such functions on Asplund spaces, the Clarke and the Mordukhovich limiting subdifferentials coincide.

2 Preliminaries

Throughout the paper, unless otherwise stated, X is a real Banach space and X^* its topological dual. The closed unit ball of X centered at zero will be denoted by $\mathbb{B}_X$, and $B[x,r]$ (resp $B(x,r)$) is the closed (resp open) ball of radius $r > 0$ centered at the point x of X.

The development of the paper is based on three basic subdifferentials: The Clarke subdifferential, the Fréchet subdifferential, and the Mordukhovich limiting subdifferential. One of the best ways to define the Clarke subdifferential is through the Clarke tangent and normal cones. Let S be a subset of X and $\overline{x} \in S$. The *Clarke tangent cone* of S at $\overline{x}$ is defined as the Painlevé-Kuratowski limit inferior of the set-differential quotient

$$T^C(S,\overline{x}) := \operatorname*{Lim\,inf}_{t\downarrow 0, x \xrightarrow[S]{} \overline{x}} \frac{1}{t}(S - x),$$

which means that $h \in T^C(S,\overline{x})$ provided, for any sequence $(t_n)_n$ tending to 0 with $t_n > 0$ and $(x_n)_n$ converging to $\overline{x}$ with $x_n \in S$, there exists a sequence $(h_n)_n$ converging to h with $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$ (see [9]). The *Clarke normal cone* $N^C(S,\overline{x})$ of S at $\overline{x}$ is the negative polar $(T^C(S,\overline{x}))^\circ$ of the Clarke tangent cone, that is,

$$N^C(S,\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq 0, \forall h \in T^C(S,\overline{x})\}.$$

Now let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $\overline{x} \in \operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The *Clarke subdifferential* $\partial_C f(\overline{x})$ of f at $\overline{x}$ is then defined as

$$\partial_C f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in N^C(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$ is the epigraph of f. When $x \notin \operatorname{dom} f$, one puts by convention $\partial_C f(x) = \emptyset$. The effective domain of the set-valued mapping $\partial_C f$ is denoted by $\mathrm{Dom}\,\partial_C f$, where for a set-valued mapping $M : X \rightrightarrows X^*$,

$$\mathrm{Dom}\, M := \{x \in X : \, M(x) \neq \emptyset\,\}.$$

If $g : X \to \mathbb{R}$ is locally Lipschitz continuous near $\overline{x}$, the Clarke subdifferential enjoys the sum rule

$$\partial_C(f+g)(\overline{x}) \subset \partial_C f(\overline{x}) + \partial_C g(\overline{x}) \tag{2.1}$$

as well as the chain rule

$$\partial_C(g \circ G)(\overline{y}) \subset A^*\big(\partial_C g(\overline{x})\big) := \{A^*(x^*) : x^* \in \partial_C g(\overline{x})\}, \tag{2.2}$$

whenever $G : Y \to X$ is a mapping from a Banach space Y into X of class C^1 near $\overline{y} \in G^{-1}(\overline{x})$ and A is the derivative of G at $\overline{y}$, that is, $A := DG(\overline{y})$; see [9]. By A^* in (2.2) we denote the adjoint of $A : Y \to X$, that is, the continuous linear mapping $A^* : X^* \to Y^*$ defined by $A^*(x^*) = x^* \circ A$. It is also worth pointing out that, if γ is a Lipschitz constant of g near $\overline{x}$, one has $\partial_C g(\overline{x}) \subset \gamma\mathbb{B}_{X^*}$. Further, $\partial_C f(\overline{x}) = \{Df(\overline{x})\}$ whenever f is finite and continuously differentiable near $\overline{x}$.

Before defining the Mordukhovich limiting subdifferential, let us consider the Fréchet subdifferential. An element $x^* \in X^*$ is a *Fréchet subgradient* of f at $\overline{x} \in \mathrm{dom}\, f$ if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\langle x^*, x - \overline{x}\rangle \leq f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\| \text{ for all } x \in B(\overline{x}, \delta).$$

The set of all Fréchet subgradients of f at $\overline{x}$ is called the *Fréchet subdifferential* of f at $\overline{x}$ and is denoted by $\partial_F f(\overline{x})$; as above one puts $\partial_F f(x) = \emptyset$ whenever f is not finite at x. *The Mordukhovich limiting subdifferential* $\partial_L f(\overline{x})$ of f at $\overline{x}$ is then defined as the set of $x^* \in X^*$ for which there exist a sequence $(x_n)_n$ in X with $(x_n, f(x_n)) \to (\overline{x}, f(\overline{x}))$ and a sequence $(x_n^*)_n$ in X^* converging weakly* to x^* with $x_n^* \in \partial_F f(x_n)$ for all $n \in \mathbb{N}$. It will be convenient as usual to write $x_n \to_f \overline{x}$ in place of $(x_n, f(x_n)) \to (\overline{x}, f(\overline{x}))$. It is known that

$$\partial_F f(x) \subset \partial_L f(x) \subset \partial_C f(x);$$

further, when f is convex the three subdifferentials coincide with the Fenchel subdifferential of convex analysis, that is, with the set

$$\{x^* \in X^* : \, \langle x^*, x - \overline{x}\rangle \leq f(x) - f(\overline{x}), \ \forall x \in X\}. \tag{2.3}$$

The Mordukhovich limiting subdifferential enjoys full calculus whenever the Banach space X is Asplund, that is, the topological dual of any separable subspace of X is separable. If the Banach spaces X and Y are Aspund, and if the above functions $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R}$ are lower semicontinuous near $\overline{x}$ and Lipschitz continuous near $\overline{x}$ respectively, then the sum rule

$$\partial_L(f+g)(\overline{x}) \subset \partial_L f(\overline{x}) + \partial_L g(\overline{x}) \tag{2.4}$$

as well as the chain rule

$$\partial_L(g \circ G)(\overline{y}) \subset A^*\big(\partial_L g(\overline{x})\big) \tag{2.5}$$

hold true, where $\overline{y} \in G^{-1}(\overline{x})$ and the mapping G and the linear mapping A are as above; see [26, 27].

The Clarke subdifferential is connected with the Mordukhovich limiting subdifferential and with the horizon limiting subdifferential $\partial_L^\infty f(\overline{x})$ which is defined as the set of $x^* \in X^*$ for which there are a sequence $(t_n)_n$ tending to 0 with $t_n > 0$, a sequence $(x_n)_n$ in X with $x_n \to_f \overline{x}$, and a sequence $(x_n^*)_n$ in X^* with $x_n^* \in \partial_F f(x_n)$ along with $(t_n x_n^*)_n$ converging weakly* to x^*. If X is an Asplund space and if f is lower semicontinuous near $\overline{x}$, then (see [18, 26])

$$\partial_C f(\overline{x}) = \overline{\mathrm{co}}^*\big(\partial_L f(\overline{x}) + \partial_L^\infty f(\overline{x})\big) \tag{2.6}$$

(where $\overline{\mathrm{co}}^*$ denotes the weak* closed convex hull), and hence

$$\partial_C f(\overline{x}) = \overline{\mathrm{co}}^*\big(\partial_L f(\overline{x})\big)$$

whenever f is Lipschitz continuous near $\bar{x}$ since in such a case $\partial_L^\infty f(\overline{x}) = \{0\}$ as easily seen.

3 s-Lower regular functions

Definition 3.1. For a real $s > 0$, we say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *s-lower regular on an open convex set $\mathcal{O}$* of the Banach space X with $\mathcal{O} \cap \mathrm{dom} f \neq \emptyset$, when it is lower semicontinuous on $\mathcal{O}$ and there exists some real coefficient $c \geq 0$ such that for all $x \in \mathcal{O} \cap \mathrm{Dom}\, \partial_C f$ and for all $x^* \in \partial_C f(x)$ we have

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c\,(1 + \|x^*\|)\,\|x - y\|^{s+1},\ \forall y \in \mathcal{O}. \tag{3.1}$$

The real constant $c \geq 0$ is called a *coefficient of s-lower regularity* of f on $\mathcal{O}$.

When $s = 1$, the function f is just said to be *primal lower regular* or *primal lower nice* on $\mathcal{O}$. $\square$

Remark. If the inequality (3.1) holds with a real $s \geq 1$ for all y in a neighborhood of a point $\overline{x} \in \operatorname{dom} f$, then it also holds with $s = 1$ for all y in some neighborhood of $\overline{x}$ (as easily seen). So, f is primal lower regular near a point $\overline{x} \in \operatorname{dom} f$ whenever it is s-lower regular near $\overline{x} \in \operatorname{dom} f$ with some real $s \geq 1$. $\square$

Of course, proper lower semicontinuous convex functions are s-lower regular on X since Clarke and Fenchel subdifferentials coincide for convex functions (as recalled in the previous section). Assuming that f is finite and Fréchet differentiable on the open convex set $\mathcal{O}$ and that the derivative Df is Hölder continuous on $\mathcal{O}$ with exponent s and coefficient $c \geq 0$, then for all $x, y \in \mathcal{O}$ we have

$$\begin{aligned} f(x) - f(y) &= \langle Df(x), x - y \rangle + \int_0^1 \langle Df(y + t(x - y)) - Df(x), x - y \rangle \, dt \\ &\leq \langle Df(x), x - y \rangle + c\|x - y\|^{1+s} \int_0^1 (1 - t)^s \, dt, \end{aligned}$$

and the latter inequality is equivalent to

$$f(y) \geq f(x) + \langle Df(x), y - x \rangle - \frac{c}{s+1}\|x - y\|^{s+1}.$$

Since $\partial_C f(x) = \{Df(x)\}$ (see the previous section), we see that the function f is s-lower regular on $\mathcal{O}$ whenever the derivative of f exists on $\mathcal{O}$ and is H́oderian therein with exponent s.

Other examples of s-lower regular functions are some convexly composite functions as established in the next proposition using the main idea in the above arguments. Let $G : X \to Y$ be a mapping which is continuously differentiable on an open convex set $\mathcal{O}$ of X and let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The *convexly composite function* $g \circ G$ is said to be *qualified* at $\overline{x} \in \mathcal{O} \cap G^{-1}(\operatorname{dom} g)$ whenever the following Robinson qualification condition holds:

$$\mathbb{R}_+\big(\operatorname{dom} g - G(\overline{x})\big) - DG(\overline{x})(X) = Y, \tag{3.2}$$

where as usual $\mathbb{R}_+ := [0, +\infty[$. For such a qualified convexly composite function it is known (see, e.g., [11]) that

$$\partial_C(g \circ G)(\overline{x}) = A^*\big(\partial_C g(\overline{y})\big), \quad \text{where } \overline{y} := G(\overline{x}) \text{ and } A := DG(\overline{x}).$$

As above A^* is the adjoint of A, hence here $A^*(y^*) = y^* \circ A$ for all $y^* \in Y^*$. It is also known that (when it holds) the Robinson qualification condition at $\overline{x}$ above is preserved for all x in a neighborhood of $\overline{x}$, and that there exist (see [11]) a neighborhood $U \subset \mathcal{O}$ of $\overline{x}$ and two real numbers $\rho > 0$ and $\sigma > 0$ such that

$$\sigma\mathbb{B}_Y \subset \big(\{g \leq \rho + g\big(G(x)\big)\} - G(x)\big) - DG(x)(\rho\mathbb{B}_X) \tag{3.3}$$

for all $x \in U \cap G^{-1}(\operatorname{dom} g)$, where $\{g \leq r\} := \{y \in Y : g(y) \leq r\}$.

Let us say that $g \circ G$ is a *qualified convexly $\mathcal{C}^{1,s}$-composite function* on $\mathcal{O}$ whenever it is a qualified convexly composite function on $\mathcal{O}$ and the mapping G is of class $\mathcal{C}^{1,s}$ on $\mathcal{O}$. Recall that G is of class $\mathcal{C}^{1,s}$ on $\mathcal{O}$ provided it is Fréchet differentiable on $\mathcal{O}$ and its derivative DG is locally Hölder continuous on $\mathcal{O}$ with exponent s. Using the idea in the above arguments (when f itself is $C^{1,s}$) and adapting some ideas in [2], we prove that qualified convexly $\mathcal{C}^{1,s}$-composite functions are s-lower regular.

Proposition 3.1. *Let $f = g \circ G$ be a qualified convexly $C^{1,s}$-composite function at $\overline{x} \in \operatorname{dom} f$. Then f is s-lower regular near $\overline{x}$; further there exists a neighborhood U of $\overline{x}$, such that for every $r > 0$ the restriction of f to $\operatorname{proj}_X\big((U \times r\mathbb{B}_{X^*}) \cap \operatorname{gph} \partial_C f\big)$ is Lipschitz continuous.*

Proof. By assumptions there exist a real $c > 0$ and a convex open neighborhood U of $\overline{x}$ over which G is c-Lipschitzian and DG is Hölderian with exponent s and coefficient c. We can also suppose according to (3.3) that there are some reals $\sigma > 0$ and $\rho > 0$ such that

$$\sigma\mathbb{B}_Y \subset \big(\{g \leq \rho + g\big(G(x)\big)\} - G(x)\big) - DG(x)(\rho\mathbb{B}_X),$$

for all $x \in U \cap G^{-1}(\operatorname{dom} g)$. Fix any $x \in U \cap \operatorname{Dom} \partial_C f$ and $x^* \in \partial_C f(x)$. By what precedes, we know that there is some $y^* \in \partial_C g(G(x))$ such that $x^* = y^* \circ DG(x)$. Fix any $y \in \mathbb{B}_Y$ and choose, according to the latter inclusion above, $b \in \mathbb{B}_X$ and $z \in Y$ with $g(z) \leq \rho + g(G(x))$ such that $\sigma y = z - G(x) - \rho DG(x)(b)$. It follows from this and the Fenchel inequality characterization of $\partial_C g$ that

$$\begin{aligned}\langle y^*, \sigma y\rangle = \langle y^*, z - G(x)\rangle - \rho\langle y^*, DG(x)(b)\rangle &\leq g(z) - g(G(x)) - \rho\langle y^*, DG(x)(b)\rangle \\ &\leq \rho - \rho\langle x^*, b\rangle \leq \rho + \rho\|x^*\|,\end{aligned}$$

which ensures that

$$\sigma\|y^*\| \le \rho(1+\|x^*\|). \tag{3.4}$$

Let us prove the s-lower regularity of f. For any $u \in U$ we can write

$$\begin{aligned} f(x)-f(u) &\le \langle y^*, G(x)-G(u)\rangle \\ &= \langle y^*, DG(x)(x-u) + \int_0^1 [DG(u+t(x-u)) - DG(x)](x-u)\,dt\rangle \\ &= \langle x^*, x-u\rangle + \langle y^*, \int_0^1 [DG(u+t(x-u)) - DG(x)](x-u)\,dt\rangle. \end{aligned}$$

From this and (3.4) we see that

$$f(x)-f(u) \le \langle x^*, x-u\rangle + \sigma^{-1}\rho c(1+\|x^*\|)\|x-u\|^{s+1}\int_0^1 (1-t)^s\,dt,$$

which means that

$$f(u) \ge f(x) + \langle x^*, u-x\rangle - \frac{c\rho}{\sigma(s+1)}(1+\|x^*\|)\|x-u\|^{s+1},$$

which translates the s-lower regularity of f on U.

It remains to show the Lipschitz property. Fix any real $r>0$ and suppose that $x^* = y^*\circ DG(x)$ is in $r\mathbb{B}_{X^*}$. Then for every (u,u^*) in $(U\times r\mathbb{B}_{X^*}) \cap \operatorname{gph}\partial_C f$, we have by the convexity of g and by (3.4)

$$\begin{aligned} f(x)-f(u) = g(G(x)) - g(G(u)) &\le \langle y^*, G(x)-G(u)\rangle \\ &\le \sigma^{-1}c\rho(1+r)\|x-u\|. \end{aligned}$$

This and the symmetry between (x,x^*) and (u,u^*) guarantees the desired Lipschitz property of the proposition and finishes its proof. $\square$

Remark. Instead of s-lower regularity with a real $0<s\le 1$, we could deal with the concept of $\omega(\cdot)$-lower regularity where $\omega : [0,+\infty[\to [0,+\infty[$ is a nondecreasing upper semicontinuous function with $\omega(0)=0$. Define a lower semicontinuous function $f : X \to \mathbb{R}\cup\{+\infty\}$ as $\omega(\cdot)$-lower regular near $\overline{x} \in \operatorname{dom} f$ provided there are an open convex neighborhood $\mathcal{O}$ of $\overline{x}$ and a real $c>0$ such that

$$f(y) \ge f(x) + \langle x^*, y-x\rangle - c(1+\|x^*\|)\|x-y\|\omega(\|x-y\|)$$

for all $y \in U$, $x \in U \cap \mathrm{Dom}\, \partial_C f$, and $x^* \in \partial_C f(x)$. Concerning such a concept, it can be proved as above that any convexly composite function $g \circ G$ is $\omega(\cdot)$-lower regular near $\overline{x}$ whenever the mapping G is of class $C^{1,\omega(\cdot)}$ on an open convex neighborhood U of $\overline{x}$, that is, the derivative DG is uniformly continuous on U and the modulus of continuity of DG relative to U is bounded from above by $r\omega(\cdot)$ for some real constant $r > 0$. Statements similar to the next results below can also be obtained in that framework. We did not develop that viewpoint since the study of differentiability properties in the other paper [17] seem to be more natural with Moreau s-envelope of s-lower regular functions than the Moreau $\omega(\cdot)$-envelope of $\omega(\cdot)$-lower regular ones. $\square$

The following first theorem shows that the concept of lower regularity is invariant with respect to the above three subdifferentials provided that the space X is Asplund.

Theorem 3.1. *Let X be an Asplund space, $\mathcal{O}$ be an open convex set of X, and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function which is lower semicontinuous on $\mathcal{O}$ with $\mathcal{O} \cap \mathrm{dom}\, f \neq \emptyset$. Let a real $s > 0$. Then the following assertions hold:*

(a) If f is s-lower regular on $\mathcal{O}$, then for all $x \in \mathcal{O}$, one has

$$\partial_F f(x) = \partial_L f(x) = \partial_C f(x).$$

(b) The function f is s-lower regular on $\mathcal{O}$ with coefficient $c \geq 0$ if and only if for all $x \in \mathcal{O} \cap \mathrm{Dom}\ \partial_F f$ and for all $x^* \in \partial_F f(x)$ one has

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c(1 + \|x^*\|)\|x - y\|^{s+1},\ \forall y \in \mathcal{O}.$$

(c) The function f is s-lower regular on $\mathcal{O}$ with coefficient $c \geq 0$ if and only if for all $x \in \mathcal{O} \cap \mathrm{Dom}\ \partial_L f$ and for all $x^* \in \partial_L f(x)$ one has

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c(1 + \|x^*\|)\|x - y\|^{s+1},\ \forall y \in \mathcal{O}.$$

Proof.

(a) Concerning the assertion (a) we only need to show that $\partial_C f(x) \subset \partial_F f(x)$ for all $x \in \mathcal{O}$. Let $x \in \mathcal{O} \cap \mathrm{Dom}\, \partial_C f$ and $x^* \in \partial_C f(x)$. Since f is s-lower regular on $\mathcal{O}$, there exists a real $c \geq 0$ such that

$$f(y) + c(1 + \|x^*\|)\|x - y\|^{s+1} \geq f(x) + \langle x^*, y - x\rangle, \forall y \in \mathcal{O}.$$

Fix any $\varepsilon > 0$. We can choose a real $\delta > 0$ such that $B(x, \delta) \subset \mathcal{O}$ and $c(1 + \|x^*\|)(\delta)^s \leq \varepsilon$. It results that

$$f(y) + \varepsilon\|x - y\| \geq f(x) + \langle x^*, y - x\rangle, \ \forall y \in B(x, \delta),$$

and this translates the inclusion $x^* \in \partial_F f(x)$.

(b) Only the implication ($\Leftarrow$) needs to be proved since the reverse one is obvious. Take a real $c \geq 0$ such that for all $x \in \mathcal{O} \cap \operatorname{Dom} \partial_F f$ and for all $x^* \in \partial_F f(x)$

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c(1 + \|x^*\|)\|x - y\|^{s+1}, \forall y \in \mathcal{O}. \qquad (3.5)$$

Step I. First, let us prove that $\partial_F f(x)$ is weakly* closed for any $x \in \mathcal{O}$. Fix any real $r > 0$. Suppose that $\partial_F f(x) \cap r\mathbb{B}_{X^*}$ is nonempty. Let $(x_i^*)_{i\in I}$ be a net in $\partial_F f(x) \cap r\mathbb{B}_{X^*}$ converging weakly* to $x^* \in X^*$. The weak* lower semicontinuity of dual norm $\|\cdot\|$ in X^* ensures that

$$\|x^*\| \leq \liminf_{i\in I} \|x_i^*\| \leq r.$$

On the other hand, from (3.5) we see that for all $y \in \mathcal{O}$

$$\langle x_i^*, y - x\rangle \leq f(y) - f(x) + c(1 + r)\|x - y\|^{s+1},$$

hence taking the limit we obtain

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + c(1 + r)\|x - y\|^{s+1}.$$

From this inequality, it is easily seen as in the proof of (a) that $x^* \in \partial_F f(x)$ thus $x^* \in \partial_F f(x) \cap r\mathbb{B}_{X^*}$. Consequently, for any $x \in \mathcal{O}$, the set $\partial_F f(x) \cap r\mathbb{B}_{X^*}$ is weakly* closed for all reals $r > 0$, so the convex set $\partial_F f(x)$ is weakly* closed in X^* according to the Banach-Dieudonné theorem.

Step II. Fix any $x \in \mathcal{O} \cap \operatorname{Dom} \partial_C f$ and define (see the pevious section) the sets

$$V := \{{}^{w^*}\lim x_n^* : x_n^* \in \partial_F f(x_n), x_n \underset{n\to\infty}{\longrightarrow_f} x\} = \partial_L f(x),$$

and

$$V_0 := \{{}^{w^*}\lim \sigma_n x_n^* : x_n^* \in \partial_F f(x_n), x_n \underset{n\to\infty}{\longrightarrow_f} x, \sigma_n \downarrow 0\} = \partial_L^\infty f(x),$$

where ${}^{w^*}\lim x_n^*$ denotes the limit of $(x_n^*)_n$ with respect to the weak* topology of X^*. Fix any $x^* \in V$, so there exist a sequence $(x_n)_n$ such that $(x_n, f(x_n))_n$ converges strongly to $(x, f(x))$, and a sequence $(x_n^*)_n$ in X^* converging weakly* to x^* with $x_n^* \in \partial_F f(x_n)$. The sequence $(x_n^*)_n$ is then bounded, say there exists a real $\gamma > 0$ such that $\|x_n^*\| \leq \gamma$ for all n. Further, for n large enough $x_n \in \mathcal{O}$, so from (3.5) we have

$$f(y) \geq f(x_n) + \langle x_n^*, y - x_n \rangle - c(1 + \|x_n^*\|)\|x_n - y\|^{s+1}, \forall y \in \mathcal{O},$$

hence

$$f(y) \geq f(x_n) + \langle x_n^*, y - x_n \rangle - c(1 + \gamma)\|x_n - y\|^{s+1}, \forall y \in \mathcal{O}.$$

Taking the limit as $n \to \infty$, we get

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - c(1 + \gamma)\|x - y\|^{s+1}, \forall y \in \mathcal{O}. \tag{3.6}$$

Consider now an arbitrary element $x_0^* \in V_0$, so there exist a sequence $(x_n)_n$ such that $(x_n, f(x_n))_n$ converges strongly to $(x, f(x))$, a sequence $(\sigma_n)_n$ in $]0, +\infty[$ with $\sigma_n \to 0$ and a sequence $(x_n^*)_n$ in X^* with $(\sigma_n x_n^*)_n$ converging weakly* to x_0^* with $x_n^* \in \partial_F f(x_n)$ for all n. Let a real $\gamma' > 0$ such that $\|\sigma_n x_n^*\| \leq \gamma'$ for all n. For n large enough, $x_n \in \mathcal{O}$ thus from (3.5)

$$f(y) \geq f(x_n) + \langle x_n^*, y - x_n \rangle - c(1 + \|x_n^*\|)\|x_n - y\|^{s+1}, \forall y \in \mathcal{O},$$

and this implies

$$\sigma_n f(y) \geq \sigma_n f(x_n) + \langle \sigma_n x_n^*, y - x_n \rangle - \sigma_n c(1 + \|x_n^*\|)\|x_n - y\|^{s+1}, \forall y \in \mathcal{O}.$$

This ensures that, for n large enough,

$$\sigma_n f(y) \geq \sigma_n f(x_n) + \langle \sigma_n x_n^*, y - x_n \rangle - \sigma_n c\|x_n - y\|^{s+1} - \gamma' c\|x_n - y\|^{s+1}, \forall y \in \mathcal{O},$$

so taking the limit as $n \to \infty$ gives

$$0 \geq \langle x_0^*, y - x \rangle - c\gamma'\|x - y\|^{s+1}, \forall y \in \mathcal{O} \cap \operatorname{dom} f.$$

The latter inequality and (3.6) yield

$$f(y) \geq f(x) + \langle x^* + x_0^*, y - x \rangle - c(1 + \gamma + \gamma')\|x - y\|^{s+1}, \forall y \in \mathcal{O} \cap \operatorname{dom} f,$$

so for $c' := c(1+\gamma+\gamma')$ (depending on x^* and x_0^*)

$$f(y) \geq f(x) + \langle x^* + x_0^*, y - x\rangle - c'\|x-y\|^{s+1}, \forall y \in \mathcal{O}.$$

From this inequality it is easily seen as in (a) that $x^* + x_0^* \in \partial_F f(x)$, hence $V + V_0 \subset \partial_F f(x)$. Since $\partial_F f(x)$ is convex and weakly* closed according to Step I, it results that

$$\overline{\mathrm{co}}^*[V + V_0] \subset \partial_F f(x) \subset \partial_C f(x).$$

These inclusions and the equality $\partial_C f(x) = \overline{\mathrm{co}}^*[V + V_0]$ (see (2.6)) entails that $\partial_C f(x) = \partial_F f(x)$ for all $x \in \mathcal{O}$. Consequently, the inequality (3.5) holds for any $(x, x^*) \in \operatorname{gph} \partial_C f$ with $x \in \mathcal{O}$, which means that f is s-lower regular in the sense of Definition 3.1.

(c) Since $\partial_F f(x) \subset \partial_L f(x) \subset \partial_C f(x)$, for all $x \in \mathcal{O}$, the assertion (c) follows from (a) and (b). $\square$

The second theorem of the paper characterizes the s-lower regularity of a function in terms of a property of hypomonotonicity of its subdifferential. Let us first establish the following lemma which is inspired from a similar lemma from [34] (see also [25]).

Lemma 3.1. *Let X be a normed vector space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function with $f(\overline{x}) < +\infty$. Let r be a positive number such that f is bounded from below over $B[\overline{x}, r]$ by some real α. Let $s > 0$, $\beta \in \mathbb{R}$ and θ be a nonnegative number. For each real $c \geq 0$, let*

$$F_{\beta,c}(x^*, x, y) := f(y) + \beta\langle x^*, x - y\rangle + c(1 + \|x^*\|)\|x-y\|^{s+1},$$

for all $x, y \in X$ and $x^* \in X^*$. Let any real

$$c_0 \geq \frac{|\beta| 4^{s+1}}{(2^{s+1}-1)r^s} \quad \text{such that} \quad c_0 > \frac{4^{s+1}}{(2^{s+1}-1)r^{s+1}}(f(\overline{x}) + \theta - \alpha).$$

Then, for any real $c \geq c_0$, for any $x^* \in X^*$ and for any $x \in B[\overline{x}, \frac{r}{4}]$, every point $u \in B[\overline{x}, r]$ such that

$$F_{\beta,c}(x^*, x, u) \leq \inf_{y \in B[\overline{x}, r]} F_{\beta,c}(x^*, x, y) + \theta$$

must belong to $B(\overline{x}, \frac{3r}{4})$.

Proof. Fix $s > 0$, $x \in B[\overline{x}, \frac{r}{4}]$, $x^* \in X^*$, and fix also any real $c \geq c_0$. Take any $y \in B[\overline{x}, r]$ with $\|y - \overline{x}\| \geq \frac{3r}{4}$. Since

$$\|x - y\| \geq \|\overline{x} - y\| - \|x - \overline{x}\| \geq \frac{r}{2},$$

we observe that

$$\|x - y\|^{s+1} - \|x - \overline{x}\|^{s+1} \geq \frac{2^{s+1} r^{s+1}}{4^{s+1}} - \frac{r^{s+1}}{4^{s+1}} = \frac{(2^{s+1} - 1) r^{s+1}}{4^{s+1}}.$$

Then, for $F(y) := F_{\beta,c}(x^*, x, y)$ we have

$$\begin{aligned}
&F(y) - F(\overline{x}) - \theta \\
&\geq f(y) - f(\overline{x}) - \theta + \beta\langle x^*, \overline{x} - y\rangle + c(1 + \|x^*\|)(\|x - y\|^{s+1} - \|x - \overline{x}\|^{s+1}) \\
&\geq \alpha - f(\overline{x}) - \theta - r|\beta|\|x^*\| + c(1 + \|x^*\|)\frac{(2^{s+1} - 1) r^{s+1}}{4^{s+1}} \\
&= \big(\alpha - f(\overline{x}) - \theta + c\frac{(2^{s+1} - 1) r^{s+1}}{4^{s+1}}\big) + r\|x^*\|\big(c\frac{(2^{s+1} - 1) r^{s}}{4^{s+1}} - |\beta|\big)
\end{aligned}$$

so, for $\eta := \alpha - f(\overline{x}) - \theta + c\frac{(2^{s+1}-1) r^{s+1}}{4^{s+1}} > 0$, we obtain $F(y) - \eta \geq F(\overline{x}) + \theta$, which finishes the proof of the lemma. □

Theorem 3.2. *Let $s > 0$ and $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on the Banach space X which is finite at $\overline{x}$ and lower semicontinuous near $\overline{x}$. The following are equivalent:*

(a) The function f is s-lower regular near $\overline{x}$;

(b) There exist reals $\varepsilon > 0$ and $c \geq 0$ such that for all $x_i^* \in \partial_C f(x_i)$ with $\|x_i - \overline{x}\| < \varepsilon$, $i = 1, 2$, one has

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -c(1 + \|x_1^*\| + \|x_2^*\|)\|x_1 - x_2\|^{s+1}.$$

If in addition X is an Asplund space, then each of the two following assertions is also equivalent to the s-lower regularity of the function f near the point $\overline{x}$:

(c) The inequality in (b) is fulfilled with $\partial_L f$ in place of $\partial_C f$;

(d) The inequality in (b) is fulfilled with $\partial_F f$ in place of $\partial_C f$.

Proof. First, we show that $(a) \Longrightarrow (b)$. Suppose that f is s-lower regular on some ball $B(\overline{x},\varepsilon)$ with some coefficient $c \geq 0$. Then, for $x_i \in X$ with $\|x_i - \overline{x}\| < \varepsilon$ and $x_i^* \in \partial_C f(x_i), i = 1,2$, we have by Definition 3.1

$$\begin{array}{rcl} f(x_1) & \geq & f(x_2) + \langle x_2^*, x_1 - x_2\rangle + c(1+\|x_2^*\|)\|x_1 - x_2\|^{s+1} \\ f(x_2) & \geq & f(x_1) + \langle x_1^*, x_2 - x_1\rangle + c(1+\|x_1^*\|)\|x_1 - x_2\|^{s+1}, \end{array}$$

and adding these inequalities we obtain

$$\begin{array}{rcl} \langle x_1^* - x_2^*, x_1 - x_2\rangle & \geq & -c(2+\|x_1^*\|+\|x_2^*\|)\|x_1 - x_2\|^{s+1} \\ & \geq & -2c(1+\|x_1^*\|+\|x_2^*\|)\|x_1 - x_2\|^{s+1}. \end{array}$$

So, the implication $(a) \Rightarrow (b)$ holds true.

Let us prove the reverse implication. Let $\varepsilon > 0, c \geq 0$ be such that the assertion (b) is fulfilled and f is lower semicontinuous on $B(\overline{x},\varepsilon)$. Let $0 < \varepsilon' < \min\{\varepsilon, \frac{1}{c^{1/s}}\}$ be such that $\alpha := \inf\limits_{B[\overline{x},\varepsilon']} f$ is finite (according to the lower semicontinuity property of f). We fix a real $c_0 \geq \frac{4^{s+1}}{(2^{s+1}-1)(\varepsilon')^s}$ with $c_0 > \frac{4^{s+1}}{(2^{s+1}-1)(\varepsilon')^{s+1}}\left(f(\overline{x}) + 1 - \alpha\right)$ and a real $c' > \max\left\{c_0, \frac{2c}{s+1-(s+1)c(\varepsilon')^s}\right\}$. Let $x \in \operatorname{Dom} \partial_C f \cap B(\overline{x}, \frac{\varepsilon'}{4})$ and $x^* \in \partial_C f(x)$. We define

$$\varphi(y) := f(y) + \langle x^*, x - y\rangle + c'\left(1+\|x^*\|\right)\|y - x\|^{s+1}, \text{ for all } y \in X$$

and

$$\overline{\varphi}(y) := \begin{cases} \varphi(y) & \text{if} \quad y \in B[\overline{x},\varepsilon'] \\ +\infty & \text{if} \quad y \in X \setminus B[\overline{x},\varepsilon'], \end{cases} \tag{3.7}$$

so clearly $\overline{\varphi}$ is lower semicontinuous on X (since f is lower semicontinuous on $B[\overline{x},\varepsilon']$).

Let (ε_n) be a sequence of real numbers which converges to 0 with $0 < \varepsilon_n < \min\left\{1, (\frac{\varepsilon'}{4})^2\right\}$. For every $n \in \mathbb{N}$, choose $u_n \in X$ such that

$$\overline{\varphi}(u_n) < \inf_X \overline{\varphi} + \varepsilon_n.$$

Applying the last lemma with $\beta = \theta = 1$ we obtain that $u_n \in B(\overline{x}, \frac{3\varepsilon'}{4})$ for all $n \in \mathbb{N}$. By the Ekeland variational principle (see [16]), for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$\|x_n - u_n\| \leq \sqrt{\varepsilon_n}, \overline{\varphi}(x_n) < \inf_X \overline{\varphi} + \varepsilon_n, \overline{\varphi}(x_n) = \inf_{u \in X}\left\{\overline{\varphi}(u) + \sqrt{\varepsilon_n}\|u - x_n\|\right\},$$

then

$$\|x_n - \overline{x}\| < \varepsilon' \text{ and } 0 \in \partial_C(\overline{\varphi} + \sqrt{\varepsilon_n}\| \cdot -x_n\|)(x_n).$$

Since $\varphi = \overline{\varphi}$ on $B[\overline{x}, \varepsilon']$ and $x_n \in B(\overline{x}, \varepsilon')$, we deduce $0 \in \partial_C(\varphi + \sqrt{\varepsilon_n}\| \cdot -x_n\|)(x_n)$ hence by (2.4) and the $\sqrt{\varepsilon_n}$-Lipschitz property of $\sqrt{\varepsilon_n}\| \cdot \|$

$$0 \in \partial_C f(x_n) - x^* + c'(1 + \|x^*\|)\partial_C(\| \cdot -x\|^{s+1})(x_n) + \sqrt{\varepsilon_n}\mathbb{B}_{X^*},$$

which furnishes some $x_n^* \in \partial_C f(x_n)$ and $y_n^* \in -x^* + c'(1 + \|x^*\|)\partial_C(\|. - x\|^{s+1})(x_n)$ with

$$\|x_n^* + y_n^*\| \le \sqrt{\varepsilon_n}. \tag{3.8}$$

Set $z_n^* := \frac{y_n^* + x^*}{c'(1+\|x^*\|)} \in \partial_C(\| \cdot -x\|^{s+1})(x_n)$, hence as easily seen (through the equality (2.3) with the convex function $\|. - x\|^{s+1}$)

$$\langle z_n^*, x_n - x\rangle = (s+1)\|x_n - x\|^{s+1} \text{ and } \|z_n^*\| = (s+1)\|x_n - x\|^s. \tag{3.9}$$

On the other hand,

$$\|x_n - x\| \le \|x_n - u_n\| + \|u_n - \overline{x}\| + \|\overline{x} - x\| < \sqrt{\varepsilon_n} + \frac{3\varepsilon'}{4} + \|\overline{x} - x\|$$

and $\frac{3\varepsilon'}{4} + \|\overline{x} - x\| < \varepsilon'$, so there exists some integer n_0 such that, for all $n \ge n_0$, $\|x_n - x\| < \varepsilon'$ and $\|z_n^*\| \le (s+1)(\varepsilon')^s$. Fix any $n \ge n_0$. From the equality $y_n^* = -x^* + c'(1 + \|x^*\|)z_n^*$ we see that

$$\|y_n^*\| \le \|x^*\| + c'(s+1)(\varepsilon')^s(1 + \|x^*\|),$$

and from the inequality

$$\|x_n^*\| \le \|x_n^* + y_n^*\| + \|y_n^*\|$$

and (3.8) we also see that

$$\|x_n^*\| \le \sqrt{\varepsilon_n} + \|x^*\| + c'(s+1)(\varepsilon')^s(1 + \|x^*\|). \tag{3.10}$$

Using the assertion (b) with $x_1^* = x_n^*, x_2^* = x^*$, ensures that

$$\langle x^* - x_n^*, x - x_n\rangle \ge -c\left(1 + \|x^*\| + \|x_n^*\|\right)\|x - x_n\|^{s+1}.$$

Since (3.8) and (3.9) entail

$$\begin{aligned}
\langle x^* - x_n^*, x - x_n\rangle &= \langle c'(1 + \|x^*\|)z_n^* - y_n^* - x_n^*, x - x_n\rangle \\
&= -(s+1)c'(1 + \|x^*\|)\|x_n - x\|^{s+1} + \langle -y_n^* - x_n^*, x - x_n\rangle \\
&\le -(s+1)c'(1 + \|x^*\|)\|x_n - x\|^{s+1} + \sqrt{\varepsilon_n}\|x - x_n\|,
\end{aligned}$$

it results that

$$
\begin{aligned}
&-(s+1)c'(1+\|x^*\|)\|x_n-x\|^{s+1}+\sqrt{\varepsilon_n}\|x-x_n\|\\
&\geq -c(1+\|x^*\|+\|x_n^*\|)\|x-x_n\|^{s+1},
\end{aligned}
$$

or equivalently

$$
\big((s+1)c'(1+\|x^*\|)-c(1+\|x^*\|+\|x_n^*\|)\big)\|x-x_n\|^s \leq \sqrt{\varepsilon_n}. \tag{3.11}
$$

Further, the inequality (3.10) implies

$$
\begin{aligned}
&(s+1)c'(1+\|x^*\|)-c(1+\|x^*\|+\|x_n^*\|)\\
&\geq (s+1)c'(1+\|x^*\|)-c(1+\|x^*\|)-c\big(\sqrt{\varepsilon_n}+\|x^*\|+c'(s+1)(\varepsilon')^s(1+\|x^*\|)\big)\\
&> (s+1)c'(1+\|x^*\|)-c(1+\|x^*\|)-c\big(1+\|x^*\|+c'(s+1)(\varepsilon')^s(1+\|x^*\|)\big)\\
&= (1+\|x^*\|)\big((s+1)c'-2c-(s+1)cc'(\varepsilon')^s\big),
\end{aligned}
$$

so by (3.11) we obtain

$$
(1+\|x^*\|)((s+1)c'-2c-(s+1)cc'(\varepsilon')^s)\|x-x_n\|^s \leq \sqrt{\varepsilon_n}. \tag{3.12}
$$

By the choice of c' we have

$$
c' > \frac{2c}{(s+1)-(s+1)c(\varepsilon')^s} \text{ or equivalently } (s+1)c'-2c-(s+1)cc'(\varepsilon')^s > 0,
$$

then it follows from (3.12) that

$$
\lim_{n\to\infty} x_n = x \quad \text{hence} \quad \lim_{n\to\infty} u_n = x.
$$

Further, we know that $\varphi(u_n) \leq \inf_{y\in B[\overline{x},\varepsilon']} \varphi(y)+\varepsilon_n$ or equivalently

$$
\begin{aligned}
&f(u_n)+\langle x^*, x-u_n\rangle + c'(1+\|x^*\|)\|u_n-x\|^{s+1}\\
&\leq \inf_{y\in B[\overline{x},\varepsilon']}\left\{f(y)+\langle x^*, x-y\rangle + c'(1+\|x^*\|)\|y-x\|^{s+1}\right\}+\varepsilon_n.
\end{aligned}
$$

Since f is lower semicontinuous and $\lim_{n\to\infty} u_n = x$, the latter inequality ensures that

$$
f(x) \leq \liminf_{n\to\infty} f(u_n) \leq \inf_{y\in B[\overline{x},\varepsilon']}\{f(y)+\langle x^*, x-y\rangle + c'(1+\|x^*\|)\|y-x\|^{s+1}\}
$$

and so

$$
f(x) \leq f(y)+\langle x^*, x-y\rangle + c'(1+\|x^*\|)\|y-x\|^{s+1}, \forall y \in B(\overline{x}, \frac{\varepsilon'}{4}),
$$

and this means that f is s-lower regular near $\overline{x}$. The proof of the theorem is then complete. □

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