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REGULARIZATION OF DYNAMICAL SYSTEMS ASSOCIATED WITH PROX-REGULAR MOVING SETS

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Abstract. This paper is concerned with the regularization of the dynamical differential inclusion of a perturbed sweeping process associated, on an interval, with the normal cone to a moving set $C(t)$. It is shown that when the set $C(t)$ is ρ-prox-regular and moves in a Lipschitzian way, one can regularize the differential inclusion by a family of usual differential equations which are well posed. The family of solutions of the regularized differential equations is shown to converge uniformly to a solution of the differential inclusion.

1. Introduction

Let H be a real Hilbert space and let $C : [T_0, T] \rightrightarrows H(T > T_0)$ be a set-valued mapping with nonempty closed values. The paper is concerned with the differential inclusion

$$(E_f) \begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) - f(t, u(t)) \\ u(T_0) = a \in C(T_0). \end{cases}$$

Here $f : [T_0, T] \times H \longrightarrow H$ is a mapping which is Bochner measurable with respect to the first variable and Lipschitzian with respect to the second variable; $N_{C(t)}(\cdot)$ (see the next section) denotes the Mordukhovich or basic normal cone to the closed set $C(t)$.

The differential inclusion (E_f) or some of its variants appear in modelizations in several fields as resource allocation mechanisms in economics (see, e.g., [11, 15]), complementarity systems (see [2]), dissipative systems in electrical circuits (see [16]), crowd motion modelization (see [18]), etc. See also [4, 5, 6, 12, 13, 17, 24, 29, 31] for other contributions.

When the sets $C(t)$ are supposed to be convex, J.J. Moreau [21, 22, 23] proved an existence and uniqueness result for the solution of $E_0(f \equiv 0)$. Moreau's method in [21, 22] consists in discretizing $[T_0, T]$ by an appropriate subdivision $T_0 = t_0^n < t_1^n < \cdots < t_{p-1}^n < t_p^n = T$ and in taking the iterates $u_0^n = a, u_{i+1}^n = \operatorname{proj}_{C(t_{i+1}^n)}(u_i)$, and then defining a mapping $u^n(\cdot)$ through those iterates. This is known as Moreau catching-up algorithm. This method of discretization was extended to the nonconvex case by G. Colombo and V.V. Goncharov [8], M. Bounkhel and L. Thibault [1], J. F. Edmond and L. Thibault [12]. Those authors proved the convergence of

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the Moreau catching-up algorithm under a prox-regularity assumption on the sets $C(t)$ of the Hilbert space H. For previous works with complements of open convex sets in finite dimensions, we refer to M. Valadier [31], C. Castaing, T.X. Duc Ha and M. Valadier [5].

Recently, with a regularization procedure which was used in the convex case by J.J. Moreau [22], L. Thibault [30] proved that under the prox-regularity of $C(t)$ one can regularize the problem (E_0) to obtain a usual differential equation

$$(E_{0,\lambda}) \begin{cases} \dot{u}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) \\ u_\lambda(T_0) = a \in C(T_0), \end{cases}$$

whose classical solution is shown to converge in some sense to the solution of (E_0).

M. Mazade and L. Thibault [19] used the same procedure to regularize, for a fixed local prox-regular set K, the differential inclusion

$$\begin{cases} \dot{u}(t) \in -N_K(u(t)) - f(t,u(t)) \\ u(T_0) = a \in K, \end{cases} \tag{1.1}$$

into the following differential equation

$$\begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda}\nabla d^2_K(u_\lambda(t)) - f(t,u_\lambda(t)) \\ u_\lambda(T_0) = a \in K. \end{cases} \tag{1.2}$$

The main difference between (1.1) and (E_f) is the fact that in (1.1) the set K does not depend on the time and it is locally prox-regular while in (E_f) the set $C(\cdot)$ moves in a Lipschitzian way with respect to the time.

In the present paper, our purpose is to use the same regularization process in [19] to regularize (E_f) into the following usual differential equation

$$(E_{f,\lambda}) \begin{cases} \dot{u}_\lambda(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) - f(t,u_\lambda(t)) \\ u_\lambda(T_0) = a \in C(T_0). \end{cases}$$

Indeed it is shown that, when the set $C(t)$ is ρ-prox-regular and moves in a Lipschitzian way with γ as a Lipschitz constant, there exists some real $\theta > 0$ (independent of λ) such that the regularized differential equation $(E_{f,\lambda})$ is well-posed on $[T_0, T_0+\theta]$ with a unique solution $u_\lambda(\cdot)$ on $[T_0, T_0+\theta]$ and that the family $(u_\lambda(\cdot))_\lambda$ converges uniformly on $[T_0, T_0+\theta]$ (when $\lambda \downarrow 0$) to a solution of (E_f) on $[T_0, T_0+\theta]$. The existence and uniqueness of solution of (E_f) is then derived on the whole interval $[T_0, T]$ through a division of the interval into subintervals of lenght less than θ.

The paper is organized as follows. In Section 2 we recall the main concepts and results used throughout the paper and we state the main theorem. The proof of the theorem is developed in Section 3 through several lemmas; many of them have their own interest.

2. Preliminaries and statement of the main result

Throughout, H is a *real Hilbert space* and S is a *closed subset* of H.
For $x \in H$ and $\delta > 0$, we will denote by $B(x, \delta)$ the open ball of H centered at x with radius δ.

Definition 2.1. For $x \in S$, an element $\zeta \in H$ is a Fréchet normal vector of the set S at x when for any real $\varepsilon > 0$ there exists some neighbourhood V of x such that

$$\langle \zeta, x' - x \rangle \le \varepsilon \|x' - x\| \quad \text{for all} \quad x' \in S \cap V.$$

The set $N^F(S;x)$ of all Fréchet normal vectors of S at x is called the *Fréchet normal cone* of S at x. For $x \notin S$ one put $N^F(S;x) = \emptyset$. The Fréchet normal cone is related to the Fréchet subdifferential of a function see[20, 27] in the following way.

Definition 2.2. Let U be a neighbourhood of a point x in H and let $\varphi : U \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real valued function. An element $\zeta \in H$ is a Fréchet subgradient of φ at x if $(\zeta, -1) \in N^F(\text{epi}\varphi; (x, \varphi(x))$.

The set $\partial_F \varphi(x)$ of all Fréchet subgradients of φ at x is called the *Fréchet subdifferential* of φ at x. When $\partial_F \varphi(x) \neq \emptyset$ one says that φ is Fréchet subdifferentiable at x.

The Fréchet subdifferential enjoys only fuzzy calculus rules see[20], that is, for any function $g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ which is Lipschitzian near $x \in U$, for $\zeta \in \partial_F(\varphi + g)(x)$ and $\varepsilon > 0$, there exist $x', x'' \in U$ such that

$$\zeta \in \partial_F \varphi(x') + \partial_F g(x'') + \varepsilon \mathbb{B}_H \quad \text{and} \quad \|x' - x\| + |\varphi(x') - \varphi(x)| < \varepsilon,$$

where $\mathbb{B}_H$ denotes the closed unit ball of H centered at zero. So a limiting process is needed to obtain calculus rules with the reference point. One defines the *Mordukhovich or basic subdifferential* of φ at $x \in U$ and the *Mordukhovich or basic normal cone* of S at $x \in S$ as follows

$$\partial \varphi(x) = {}^{\text{seq}}\mathop{\text{Limsup}}_{u \to_\varphi x} \partial_F \varphi(u) \quad \text{and} \quad N(S;x) = {}^{\text{seq}}\mathop{\text{Limsup}}_{S \ni u \to x} N^F(S;u).$$

The second member of the first (resp. second) equality denotes the set of all $\zeta \in H$ which are the weak limit of a sequence $(\zeta_n)_n$ with $\zeta_n \in \partial_F \varphi(u_n)$ (resp. $\zeta_n \in N^F(S;u_n)$), $(u_n)_n$ converging strongly to x, and $\varphi(u_n) \to \varphi(x)$ (resp. $u_n \in S$). It will be sometimes convenient to denote the basic normal cone of S at x by $N_S(x)$.

It is shown in [26] that the normal cone of a set and the distance function associated to this set are strongly involved in some characterizations of the prox-regularity of this set. The closed set S is (uniformly) *ρ-prox-regular* (see [26]) when any point x in the open ρ-enlargement of S

$$U_\rho(S) := \{u \in H : d_S(u) < \rho\}$$

has a unique nearest point $\text{proj}_S(x)$ in S and the mapping proj_S is continuous over $U_\rho(S)$; such sets are also called positively reached, weakly convex, p-convex, $O(2)$-convex, and proximally smooth, in [14], [32], [3] ,[28], and [7] respectively. In the definition of $U_\rho(S)$ above, $d_S(x)$, also denoted by $d(x,S)$, is the distance from the point x to the set S. A first characterization in terms of the normal cone (see [26]) is that for any nonzero vector $\zeta \in N_S(x)$ one has $x \in \text{Proj}_S(x + \frac{\rho}{\|\zeta\|}\zeta)$, where $\text{Proj}_S(y)$ is, for each $y \in H$, the set (eventually empty) of all nearest points of y in S. Translating this in the fact that for all $x' \in S$

$$\left\| x - \left(x + \frac{\rho}{\|\zeta\|}\zeta \right) \right\|^2 \le \left\| x + \frac{\rho}{\|\zeta\|}\zeta - x' \right\|^2$$

we see that this is equivalent to

$$\langle \zeta, x' - x \rangle \le \frac{\|\zeta\|}{2\rho}\|x' - x\|^2 \quad \text{for all } x' \in S. \tag{2.1}$$

This ensures that $\zeta \in N^F(S;x)$, and hence $N^F(S;x) = N(S;x)$. More generally, under the ρ-prox-regularity of the closed set S, it is known that

$$N(S;x) = N^F(S;x) = \{\zeta \in H : \exists r > 0,\ x \in \operatorname{Proj}_S(x + r\zeta)\}. \tag{2.2}$$

The inequality (2.1) implies also that

$$\langle \zeta' - \zeta, x' - x \rangle \ge -\|x' - x\|^2 \tag{2.3}$$

for all $\zeta \in N_S(x)$ and $\zeta' \in N_S(x')$ with $\|\zeta\| \le \rho$ and $\|\zeta'\| \le \rho$. In fact, in [26] it is proved that the inequality (2.3), called the *ρ-hypomonotonicity* of the truncated normal cone $N_S(\cdot) \cap \mathbb{B}_H$, characterizes the ρ-prox-regularity of the closed set S. So, in the particular case where $\rho = +\infty$, we obtain the monotonicity of the normal cone to S and hence by [10, 25] one recovers the fact that the ρ-prox-regularity of the closed set S with $\rho = \infty$ corresponds to its convexity.

A second fundamental characterization exists in terms of the differentiability of the associated distance function. In fact, it is shown in [3, 7, 26] that the following hold.

Proposition 2.3. *Let S be a nonempty closed set of the Hilbert space H. The following are equivalent:*
(a) *The set S is ρ-prox-regular;*
(b) *the squared distance function $d_S^2(\cdot)$ is continuously differentiable over $U_\rho(S)$;*
(c) *the mapping* proj_S *is well defined on $U_\rho(S)$ and, for any positive real $\delta < \rho$, the mapping* proj_S *is Lipschitzian on U_δ with $\rho/(\rho - \delta)$ as Lipschitz constant, that is, for all $x, x' \in U_\delta(S)$,*

$$\|\operatorname{proj}_S(x) - \operatorname{proj}_S(x')\| \le \frac{\rho}{\rho - \delta}\|x - x'\|. \tag{2.4}$$

Further, when S is ρ-prox-regular, one has

$$\nabla\left(\frac{1}{2}d_S^2\right)(x) = x - \operatorname{proj}_S(x) \quad \text{for all } x \in U_\rho(S). \tag{2.5}$$

Now let T_0, T be two nonnegative real numbers with $T_0 < T$ and, for each $t \in [T_0, T]$, let $C(t)$ be a nonempty closed ρ-prox-regular set in H. One says that the closed $C(t)$ moves in a Lipschitzian way with $t \in [T_0, T]$ when there exists a constant $\gamma > 0$ such that for all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \le \gamma |t - s| \tag{2.6}$$

for all $s, t \in [T_0, T]$.

We can now state the main result of the paper.

Theorem 2.4. *Assume that the closed sets $C(t)$ of the Hilbert space H are ρ-prox-regular and move in a Lipschitzian way with γ as a Lipschitz constant, that is,* (2.6) *holds. Let $a \in C(T_0)$ and let $f : [T_0, T] \times B(a, \frac{\rho}{3}) \to H$ be a mapping which is Bochner measurable with respect to $t \in [T_0, T]$ and such that:*

(i) - *there exists a real* $\beta > 0$ *such that, for all* $t \in [T_0, T]$ *and* $x \in B(a, \frac{\rho}{3})$,

$$\|f(t,x)\| \leq \beta;$$

(ii) - *there exists a non-negative real number* k *such that for all* $t \in [T_0, T]$ *and for all* $(x,y) \in B(a, \frac{\rho}{3}) \times B(a, \frac{\rho}{3})$,

$$||f(t,x) - f(t,y)|| \leq k||x-y||.$$

Let θ *be a positive real number such that* $\theta < \frac{\rho}{3(2\beta+\gamma)}$ *and let the real* $\lambda_\rho := \rho/(\beta+\gamma)$.

Under the above assumptions, for any $\lambda \in]0, \lambda_\rho[$, *the differential equation over* $[T_0, T_0+\theta] \times B(a, \frac{\rho}{3})$

$$\begin{cases} \dot{u}_\lambda(t) = (-1/2\lambda)\nabla d^2_{C(t)}(u_\lambda(t)) - f(t, u_\lambda(t)) \\ u_\lambda(T_0) = a \end{cases} \tag{2.7}$$

is well defined and has a unique solution u_λ *on* $[T_0, T_0+\theta]$, *and the family* $(u_\lambda)_{0<\lambda<\lambda_\rho}$ *converges uniformly on* $[T_0, T_0+\theta]$ *as* $\lambda \downarrow 0$ *to a solution of the dynamical differential inclusion* (E_f). *Further, this solution stays in* $B(a, \frac{\rho}{3})$ *and the solution inside this ball is unique.*

If the mapping f *is defined on* $[T_0, T] \times H$ *and satisfies the assumptions (i) and (ii) for all* $t \in [T_0, T]$ *and* $x, y \in H$, *then dividing* $[T_0, T]$ *into a finite number of intervals with length less than or equal to* θ *yields the existence of a unique solution* $u(\cdot)$ *of* (E) *over* $[T_0, T]$. *Further* $\|\dot{u}(t)\| \leq 2\beta + \gamma$ *for almost all* $t \in [T_0, T]$.

The proof of the theorem will be established in the next section through a series of lemmas.

3. Proof of the theorem

In all the lemmas below the closed sets $C(t)$ are *ρ-prox-regular* and satisfy the *inequality* (2.6).

We start the proof of Theorem 2.4 with the following lemma which is in the line of a result in [30]. For completeness we sketch the proof.

Lemma 3.1. *Let* I *be a subinterval of* $[T_0, T]$ *and* $z : I \to H$ *be a locally absolutely continuous mapping, and let* $g(t) := d(z(t), C(t))$ *for all* $t \in I$. *Assume that* $d(z(t), C(t)) < \rho$ *for all* $t \in I$. *Then, for almost every* $t \in I$,

$$\dot{g}(t)g(t) \leq \langle \dot{z}(t), z(t) - \operatorname{proj}_{C(t)}(z(t))\rangle + \gamma g(t).$$

Proof. Put $\varphi(t,x) := \frac{1}{2}d^2(x, C(t))$ for all $t \in I$ and all $x \in H$. The function g is absolutely continuous because, by (2.6),

$$|g(t) - g(s)| \leq \|z(t) - z(s)\| + \gamma|t-s|$$

for all $s, t \in I$. Fix any $t \in \operatorname{int} I :=]\tau_0, \tau_1[$ such that g and z are derivable at t. Observe that, according to Proposition 2.3 and to (2.5), the function $\varphi(t, \cdot)$ is continuously differentiable around $z(t)$ and that

$$\nabla_2\varphi(t, z(t)) = z(t) - \operatorname{proj}_{C(t)}(z(t)). \tag{3.1}$$

Write for $s \in]0, \tau_1 - t[$ small enough

$$\begin{aligned}\frac{1}{2s}[g(t+s)^2 - g(t)^2] &= \frac{1}{s}[\varphi(t+s, z(t+s)) - \varphi(t, z(t+s))] \\ &+ \frac{1}{s}[\varphi(t, z(t+s)) - \varphi(t, z(t))] \\ &\leq \frac{\gamma}{2}[d(z(t+s), C(t+s)) + d(z(t+s), C(t))] \\ &+ \frac{1}{s}[\varphi(t, z(t+s)) - \varphi(t, z(t))].\end{aligned}$$

As z is derivable at t, there exists $\varepsilon(s) \underset{s\downarrow 0}{\longrightarrow} 0$ in H such that

$$z(t+s) = z(t) + s\dot{z}(t) + s\varepsilon(s)$$

and this yields

$$\begin{aligned}\frac{1}{s}[\varphi(t, z(t+s)) - \varphi(t, z(t))] &\leq \frac{1}{s}[\varphi(t, z(t) + s\dot{z}(t)) - \varphi(t, z(t))] \\ &+ \frac{1}{2}\|\varepsilon(s)\| \, [d(z(t) + s\dot{z}(t) + s\varepsilon(s)), C(t)) \\ &+ d(z(t) + s\dot{z}(t), C(t))].\end{aligned}$$

Putting this inequality and (3) together we obtain

$$\begin{aligned}\frac{1}{2s}[g(t+s)^2 - g(t)^2] &\leq \frac{\gamma}{2}[d(z(t+s), C(t+s)) + d(z(t+s), C(t))] \\ &+ \frac{1}{s}[\varphi(t, z(t) + s\dot{z}(t)) - \varphi(t, z(t))] + \eta(s)\end{aligned}$$

for some $\eta(s) \underset{s\downarrow 0}{\longrightarrow} 0$ in $\mathbb{R}$. Taking $s \downarrow 0$, it follows that

$$\dot{g}(t)g(t) \leq \gamma g(t) + \langle \nabla_2 \varphi(t, z(t)), \dot{z}(t) \rangle$$

and the proof is complete according to (3.1). □

Throughout the remaining of the paper, fix a positive real θ satisfying

$$\theta < \frac{\rho}{3(2\beta + \gamma)}.$$

We observe that, for any $x \in B(a, \frac{\rho}{3})$ and any $t \in [T_0, T_0 + \theta]$, we have (since $a \in C(T_0)$)

$$d(x, C(t)) \leq d(a, C(T_0)) + \|x - a\| + \gamma|t - T_0| < \frac{2}{3}\rho. \tag{3.2}$$

Then, according to Proposition 2.3, for any $t \in [T_0, T_0 + \theta]$ the mapping $x \mapsto \frac{1}{2}\nabla d^2_{C(t)}(x)$ is well defined on $B(a, \frac{1}{3}\rho)$ and by (2.5)

$$\frac{1}{2}\nabla d^2_{C(t)}(x) = x - \operatorname{proj}_{C(t)}(x). \tag{3.3}$$

Further, for any $x_1, x_2 \in B(a, \frac{1}{3}\rho)$ we have, by (3.2), the inequalities

$$d(x_i, C(t)) < \frac{2}{3}\rho < \rho$$

for $i = 1, 2$, and hence taking $\delta = \frac{2}{3}\rho$, the inequality (2.4) yields

$$\|\mathrm{proj}_{C(t)}(x_1) - \mathrm{proj}_{C(t)}(x_2)\| \leq \frac{\rho}{\rho - \delta}\|x_1 - x_2\| = 3\|x_1 - x_2\|. \tag{3.4}$$

Therefore, the mapping $\frac{1}{2}\nabla d^2_{C(t)}(\cdot) + f(t, \cdot)$ is Lipschitzian over the ball $B(a, \frac{1}{3}\rho)$. On the other hand, for any $x \in B(a, \frac{1}{3}\rho)$, the mapping $\frac{1}{2}\nabla d^2_{C(\cdot)}(x) + f(\cdot, x)$ is Bochner integrable on $[T_0, T_0 + \theta]$. Further, from the assumptions of Theorem 2.4 above we have

$$\|f(t, x)\| \leq \beta, \quad \text{for all} \quad (t, x) \in [T_0, T_0 + \theta] \times B\Big(a, \frac{1}{3}\rho\Big). \tag{3.5}$$

Fix now any real number $\lambda > 0$ and consider the differential equation over $[T_0, T_0 + \theta[\times B(a, \frac{1}{3}\rho)$

$$(E^*_{f,\lambda}) \qquad \begin{cases} \dot{u_\lambda}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u_\lambda(t)) - f(t, u_\lambda(t)) \\ u_\lambda(T_0) = a. \end{cases}$$

The results just obtained above ensure us that this differential equation has a (unique) solution $u_\lambda(\cdot)$ defined on its *maximal* interval of existence $[T_0, T_\lambda[\subset [T_0, T_0 + \theta[$. In the sequel we will denote $-f(t, u_\lambda(t))$ as

$$z_\lambda(t) := -f(t, u_\lambda(t)). \tag{3.6}$$

Then by (3.3) we have

$$\dot{u}_\lambda(t) = -\frac{1}{\lambda}\left[u_\lambda(t) - \mathrm{proj}_{C(t)}(u_\lambda(t))\right] + z_\lambda(t), \text{ a.e. } t \in [T_0, T_\lambda[. \tag{3.7}$$

We recall the Gronwall lemma which will be used in the next result as well as in other places of the paper.

Lemma 3.2 (Gronwall's lemma)**.** *Let $b, c, \zeta : [t_0, t_1] \to \mathbb{R}$ be three real-valued Lebesgue integrable functions. If the function $\zeta(\cdot)$ is absolutely continuous on the interval $[t_0, t_1]$ and if for almost all $t \in [t_0, t_1]$*

$$\dot{\zeta}(t) \leq b(t) + c(t)\zeta(t),$$

then for all $t \in [t_0, t_1]$

$$\zeta(t) \leq \zeta(t_0)\exp\Big(\int_{t_0}^t c(\tau)\, d\tau\Big) + \int_{t_0}^t b(s)\exp\Big(\int_s^t c(\tau)\, d\tau\Big)\, ds.$$

Through the Gronwall lemma we can prove the following lemma which provides an upper bound for the derivative in t of the distance function from $u_\lambda(t)$ to $C(t)$.

Lemma 3.3. *Put $g_\lambda(t) := d(u_\lambda(t), C(t))$ for any $t \in [T_0, T_\lambda[$. Then g_λ is locally absolutely continuous on $[T_0, T_\lambda[$ and*

$$\dot{g}_\lambda(t) \leq (\beta + \gamma) - \frac{1}{\lambda}g_\lambda(t) \text{ a.e. } t \in [T_0, T_\lambda[.$$

Further, for all $t \in [T_0, T_\lambda[$

$$g_\lambda(t) \leq \lambda(\beta + \gamma).$$

Proof. Consider $t \in]T_0, T_\lambda[$ where $\dot{g}_\lambda(t)$ and $\dot{u}_\lambda(t)$ exist and satisfy $(E^*_{f,\lambda})$. Since $u_\lambda(t) \in B(a, \frac{1}{3}\rho)$, we see from (3.2) that $d(u_\lambda(t), C(t)) < \rho$. Taking $z(t) = u_\lambda(t)$ in Lemma 3.1, we obtain

$$\begin{aligned}\dot{g}_\lambda(t) g_\lambda(t) &\le \gamma g_\lambda(t) + \langle \dot{u}_\lambda(t), u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\rangle \\ &\le \gamma g_\lambda(t) - \frac{1}{\lambda}\langle u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t)), u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\rangle \\ &\quad + \langle z_\lambda(t), u_\lambda(t) - \operatorname{proj}_C(t)(u_\lambda(t))\rangle,\end{aligned}$$

the second inequality being due to the equality

$$\dot{u}_\lambda(t) = -\frac{1}{\lambda}\left[u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\right] + z_\lambda(t) \quad \text{according to (3.7).}$$

Since $g_\lambda(t) = d(u_\lambda(t), C(t)) = ||u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))||$, we deduce

$$\begin{aligned}\dot{g}_\lambda(t) g_\lambda(t) &\le \gamma g_\lambda(t) - \frac{1}{\lambda} g^2_\lambda(t) + \langle z_\lambda(t), u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\rangle \\ &\le \gamma g_\lambda(t) - \frac{1}{\lambda} g^2_\lambda(t) + \|z_\lambda(t)\| \|u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))\| \\ &= \gamma g_\lambda(t) - \frac{1}{\lambda} g^2_\lambda(t) + g_\lambda(t)||z_\lambda(t)||.\end{aligned}$$

Then using (3.5) and (3.6) we get

$$\dot{g}_\lambda(t) \le \beta + \gamma - \frac{1}{\lambda} g_\lambda(t) \ \text{ if } g_\lambda(t) > 0.$$

The last inequality is still valid whenever $g_\lambda(t) = 0$ because this leads to $\dot{g}_\lambda(t) = 0$. Indeed the equality $g_\lambda(t) = 0$ ensures for $|s|$ small enough that

$$\frac{1}{s}[g_\lambda(t+s) - g_\lambda(t)] = \frac{1}{s} d_{C(t)}(u_\lambda(t+s)),$$

and hence, since $\dot{g}_\lambda(t)$ exists, we deduce that

$$\dot{g}_\lambda(t) = \lim_{s\uparrow 0} \frac{1}{s} d_{C(t)}(u_\lambda(t+s)) \le 0 \text{ and } \dot{g}_\lambda(t) = \lim_{s\downarrow 0} \frac{1}{s} d_{C(t)}(u_\lambda(t+s)) \ge 0,$$

so $\dot{g}_\lambda(t) = 0$. Thus, for almost every $t \in [T_0, T_\lambda[$,

$$\dot{g}_\lambda(t) \le (\beta + \gamma) - \frac{1}{\lambda} g_\lambda(t),$$

which is the first inequality of the lemma. Further, observing that

$$g_\lambda(T_0) = d(u_\lambda(T_0), C(T_0)) = d(a, C(T_0)) = 0$$

and applying Lemma 3.2 with $b(\cdot) = \beta + \gamma$ and $c(\cdot) = -\frac{1}{\lambda}$, we also obtain for all $t \in [T_0, T_\lambda[$ that

$$\begin{aligned}g_\lambda(t) &\le e^{-t/\lambda} \int_{T_0}^t (\beta + \gamma) e^{s/\lambda}\, ds = e^{-t/\lambda}(\beta + \gamma)\lambda[e^{t/\lambda} - e^{T_0/\lambda}] \\ &= \lambda(\beta + \gamma)[1 - e^{-(t - T_0)}],\end{aligned}$$

and hence

$$g_\lambda(t) \le \lambda(\beta + \gamma).$$

□

In the next lemma we establish an upper bound of the derivative of the solution $u_\lambda(\cdot)$ of $(E^*_{f,\lambda})$.

Lemma 3.4. *For almost every $t \in [T_0, T_\lambda[$, one has*

$$||\dot{u}_\lambda(t) - z_\lambda(t)|| \le \beta + \gamma, \tag{3.8}$$

and consequently

$$||\dot{u}_\lambda(t)|| \le 2\beta + \gamma. \tag{3.9}$$

Proof. As $u_\lambda(\cdot)$ is a solution of $(E^*_{f,\lambda})$, one has

$$\dot{u}_\lambda(t) = -\frac{1}{\lambda}[u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))] + z_\lambda(t).$$

So

$$\dot{u}_\lambda(t) - z_\lambda(t) = -\frac{1}{\lambda}[u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t))].$$

This yields

$$||\dot{u}_\lambda(t) - z_\lambda(t)|| = \frac{1}{\lambda}||(u_\lambda(t) - \operatorname{proj}_{C(t)}(u_\lambda(t)))|| = (1/\lambda)g_\lambda(t),$$

and applying Lemma 3.3, it ensues that

$$||\dot{u}_\lambda(t) - z_\lambda(t)|| = (1/\lambda)g_\lambda(t) \le \beta + \gamma.$$

To obtain the last inequality in the lemma, it suffices to note that

$$||\dot{u}_\lambda(t)|| \le ||\dot{u}_\lambda(t) - z_\lambda(t)|| + ||z_\lambda(t)|| \le 2\beta + \gamma.$$

□

Lemma 3.4 tells us that the mapping $u_\lambda(\cdot)$ is $(2\beta+\gamma)$-Lipschitzian on $[T_0, T_\lambda[$. So T_λ being finite, the limit $u_\lambda(T_\lambda) := \lim_{t\uparrow T_\lambda} u_\lambda(t)$ exists in H and the extended mapping $u_\lambda(\cdot)$ is Lipschitzian on $[T_0, T_\lambda]$. Since $\theta < \frac{\rho}{3(2\beta+\gamma)}$, we have $T_\lambda - T_0 < \frac{\rho}{3(2\beta+\gamma)}$. Then for $u_\lambda(T_\lambda) := \lim_{t\uparrow T_\lambda} u_\lambda(t)$ obtained above we have

$$\|u_\lambda(T_\lambda) - a\| = \|u_\lambda(T_\lambda) - u_\lambda(T_0)\| \le (2\beta+\gamma)(T_\lambda - T_0) < \frac{1}{3}\rho \tag{3.10}$$

and hence $u_\lambda(\cdot)$ (extended at T_λ) is a Lipschitzian solution over the closed interval $[T_0, T_\lambda]$ of the differential equation

$$\begin{cases} \dot{u}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u(t)) - f(t, u(t)) \\ u(T_0) = a \end{cases}$$

with $u_\lambda([T_0, T_\lambda]) \subset B(a, \frac{\rho}{3})$. Further, $T_\lambda = T_0 + \theta$ since otherwise (3.10) would allow us to extend $u_\lambda(\cdot)$ on the right of T_λ in a solution to the differential equation $(E^*_{f,\lambda})$ with the range of the extension of $u_\lambda(\cdot)$ included in $B(a, \frac{1}{3}\rho)$, which would be in contradiction with the maximality of the interval $[T_0, T_\lambda[$.

Our analysis establishes that, for any real number $\lambda > 0$, the differential equation relative now to the closed interval $[T_0, T_0 + \theta]$ and denoted by

$$(E_{f,\lambda}) \begin{cases} \dot{u}(t) = -\frac{1}{2\lambda}\nabla d^2_{C(t)}(u(t)) - f(t, u(t)) \\ u(T_0) = a \end{cases}$$

has a unique Lipschitzian solution $u_\lambda(\cdot)$ on the whole closed interval $[T_0, T_0+\theta]$ with $u_\lambda([T_0, T_0+\theta]) \subset B(a, \frac{1}{3}\rho)$.

Next we prove that this family $(u_\lambda)_{\lambda>0}$ satisfies the cauchy criterion as $\lambda \downarrow 0$. It is mentioned above in (3.4) that $\mathrm{proj}_{C(t)}(\cdot)$ is well defined and is 3-Lipschitzian on $B(a, \rho/3)$. Also Proposition 2.3 tells us that the ρ-prox-regularity of a given closed set S is equivalent to the ρ-hypomonotonicity property of $N_S(\cdot) \cap \mathbb{B}_H$, that is,

$$\langle \zeta' - \zeta, x' - x \rangle \geq -\|x' - x\|^2$$

for all $\zeta \in N_S(x)$ and $\zeta' \in N_S(x')$ with $\|\zeta'\| \leq \rho$ and $\|\zeta\| \leq \rho$.

Further notice from (3.8) that, for $\lambda \leq \frac{\rho}{\beta+\gamma}$, we have $\|\lambda[\dot{u}_\lambda(t) - z_\lambda(t)]\| \leq \rho$. On the other hand, from (3.7) and the equality $T_\lambda = T_0 + \theta$, we can deduce that for every positive real $\lambda \leq \frac{\rho}{\beta+\gamma}$,

$$\frac{\rho}{\beta+\gamma}[-\dot{u}_\lambda(t) + z_\lambda(t)] \in N_{C(t)}\big(\mathrm{proj}_{C(t)}(u_\lambda(t))\big)$$

and the projection

$$\mathrm{proj}_{C(t)}(u_\lambda(t)) \in C(t)$$

is well defined with

$$\mathrm{proj}_{C(t)}(u_\lambda(t)) = u_\lambda(t) + \lambda[\dot{u}_\lambda(t) - z_\lambda(t)] \text{ a.e. } t \in [T_0, T_0+\theta[.$$

With those informations at hand, we prove the next lemma.

Lemma 3.5. *For all positive numbers $\lambda, \mu < \rho/(\beta+\gamma)$, one has for all $t \in [T_0, T_0+\theta]$*

$$||u_\lambda(t) - u_\mu(t)||^2 \leq 2(\lambda+\mu)(\beta+\gamma)^2 \int_{T_0}^{t} \exp\left(2\Big[9\Big(\frac{\beta+\gamma}{\rho}\Big) + k\Big](t-s)\right) ds.$$

Proof. Using the above arguments with

$$\zeta := \left(\frac{\rho}{\beta+\gamma}\right)[-\dot{u}_\lambda(t) + z_\lambda(t)] \quad \text{and} \quad \zeta' := \left(\frac{\rho}{\beta+\gamma}\right)[-\dot{u}_\mu(t) + z_\mu(t)]$$

we have for a.e. $t \in [T_0, T]$

$$\langle -\dot{u}_\lambda(t) + z_\lambda(t) + \dot{u}_\mu(t) - z_\mu(t), \mathrm{proj}_{C(t)}(u_\lambda(t)) - \mathrm{proj}_{C(t)}(u_\mu(t))\rangle$$
$$\geq -\Big(\frac{\beta+\gamma}{\rho}\Big)||\mathrm{proj}_{C(t)}(u_\lambda(t)) - \mathrm{proj}_{C(t)}(u_\mu(t))||^2.$$

According to the equality

$$\mathrm{proj}_{C(t)}(u_\lambda(t)) = \lambda(\dot{u}_\lambda(t) - z_\lambda(t)) + u_\lambda(t)$$

and according to the Lipschitzian property of $\mathrm{proj}_{C(t)}(\cdot)$ on $B(a, \rho/3)$ with 3 as a Lipschitzian constant therein, the latter inequality combined with the inclusions $u_\lambda(t), u_\mu(t) \in B(a, \rho/3)$ yields

$$\langle -\dot{u}_\lambda(t) + z_\lambda(t) + \dot{u}_\mu(t) - z_\mu(t), \lambda(\dot{u}_\lambda(t) - z_\lambda(t)) - \mu(\dot{u}_\mu(t) - z_\mu(t)) + u_\lambda(t) - u_\mu(t)\rangle$$
$$\geq -9\left(\frac{\beta+\gamma}{\rho}\right)||u_\lambda(t) - u_\mu(t)||^2.$$

Computing the left hand side, we obtain

$$-\lambda||\dot{u}_\lambda(t)-z_\lambda(t)||^2-\mu||\dot{u}_\mu(t)-z_\mu(t)||^2+(\lambda+\mu)\langle\dot{u}_\lambda(t)-z_\lambda(t),\dot{u}_\mu(t)-z_\mu(t)\rangle$$
$$+\langle z_\lambda(t)-z_\mu(t),u_\lambda(t)-u_\mu(t)\rangle-\langle\dot{u}_\lambda(t)-\dot{u}_\mu(t),u_\lambda(t)-u_\mu(t)\rangle$$
$$\geq -9\left(\frac{\beta+\gamma}{\rho}\right)||u_\lambda(t)-u_\mu(t)||^2,$$

or equivalently

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[||u_\lambda(t)-u_\mu(t)||^2]\leq 9\left[\frac{\beta+\gamma}{\rho}\right]&||u_\lambda(t)-u_\mu(t)||^2\\ &-\lambda||\dot{u}_\lambda(t)-z_\lambda(t)||^2-\mu||\dot{u}_\mu(t)-z_\mu(t)||^2\\ &+(\lambda+\mu)\langle\dot{u}_\lambda(t)-z_\lambda(t),\dot{u}_\mu(t)-z_\mu(t)\rangle\\ &+\langle z_\lambda(t)-z_\mu(t),u_\lambda(t)-u_\mu(t)\rangle.\end{aligned}$$

This entails

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[||u_\lambda(t)-u_\mu(t)||^2]\leq 9\left[\frac{\beta+\gamma}{\rho}\right]&||u_\lambda(t)-u_\mu(t)||^2\\ &+(\lambda+\mu)\langle\dot{u}_\lambda(t)-z_\lambda(t),\dot{u}_\mu(t)-z_\mu(t)\rangle\\ &+\langle z_\lambda(t)-z_\mu(t),u_\lambda(t)-u_\mu(t)\rangle.\end{aligned}$$

Concerning the second expression in the righ-hand side of the latter inequality, note by (3.8) in Lemma 3.4 that

$$||\dot{u}_\lambda(t)-z_\lambda(t)||\leq\beta+\gamma \quad\text{and}\quad ||\dot{u}_\mu(t)-z_\mu(t)||\leq\beta+\gamma.$$

Moreover, since $u_\lambda(\cdot),u_\mu(\cdot)\in B(a,\frac{1}{3}\rho)$ and $f(t,\cdot)$ is k-Lipschitzian on the ball $B(a,\frac{1}{3}\rho)$, we can write

$$\begin{aligned}||z_\lambda(t)-z_\mu(t)||&=||f(t,u_\lambda(t))-f(t,u_\mu(t))||\\ &\leq k||u_\lambda(t)-u_\mu(t)||.\end{aligned}$$

Consequently, we obtain

$$\frac{1}{2}\frac{d}{dt}[||u_\lambda(t)-u_\mu(t)||^2]\leq\left[9\left(\frac{\beta+\gamma}{\rho}\right)+k\right]||u_\lambda(t)-u_\mu(t)||^2+(\lambda+\mu)(\beta+\gamma)^2.$$

Since $u_\lambda(T_0)-u_\mu(T_0)=0$, according to Lemma 3.2, we see that for

$m:=2\left[9\left(\frac{\beta+\gamma}{\rho}\right)+k\right]$ we have

$$||u_\lambda(t)-u_\mu(t)||^2\leq 2(\lambda+\mu)(\beta+\gamma)^2\int_{T_0}^t e^{m(t-s)}\,ds,$$

which is the desired inequality of the lemma. □

The next lemma proves that the family (u_λ) converges to a solution on $[T_0,T_0+\theta]$.

Lemma 3.6. *The family* $(u_\lambda)_{0<\lambda<\frac{\rho}{\beta+\gamma}}$ *converges uniformly on* $[T_0,T_0+\theta]$ *to a solution of the differential inclusion*

$$(P)\begin{cases}\dot{u}(t)\in -N_{C(t)}(u(t))-f(t,u(t))\\ u(T_0)=a\end{cases}$$

over the interval $[T_0, T_0+\theta]$ *and* $u([T_0, T_0+\theta]) \subset B(a, \frac{\rho}{3})$.

Further, $\|\dot{u}(t) + f(t, u(t))\| \leq \beta + \gamma$ *for almost every* $t \in [T_0, T_0+\theta]$, *so* $u(\cdot)$ *is Lipschitzian over* $[T_0, T_0+\theta]$ *with* $2\beta+\gamma$ *as a Lipschitz constant.*

Proof. The above Lemma provides a uniform cauchy criterion for the family $u_\lambda(\cdot)$ on $[T_0, T_0+\theta]$ as $\lambda \downarrow 0$. Therefore $(u_\lambda(\cdot))$ converges uniformly to a continuous mapping $u(\cdot) \in \mathcal{C}([T_0, T_0+\theta], H)$ as $\lambda \downarrow 0$.

On the one hand, from Lemma3.3 we have

$$d(u_\lambda(t), C(t)) \leq \lambda(\beta+\gamma),$$

so by letting $\lambda \downarrow 0$ we get

$$u(t) \in C(t) \quad \text{for all} \quad t \in [T_0, T_0+\theta]. \tag{3.11}$$

Next, from the inequality $\|\dot{u}_\lambda(t)\| \leq 2\beta+\gamma$(see (3.9)), we can extract a sequence (λ_n), $\lambda_n \downarrow 0$, such that $\dot{u}_{\lambda_n}(\cdot)$ converges weakly in the space $L^2([T_0, T_0+\theta], H)$ to some mapping $h(\cdot) \in L^2([T_0, T_0+\theta], H)$, and $\|h(t)\| \leq 2\beta+\gamma$ a.e., since the set $\{v \in L^2([T_0, T_0+\theta], H) : \|v(t)\| \leq 2\beta+\gamma \text{ a.e.}\}$ is convex and closed in $L^2([T_0, T_0+\theta], H)$. So for any $t \in [T_0, T_0+\theta]$, fixing any $z \in H$ and writing

$$\left\langle z, \int_{T_0}^{t} \dot{u}_{\lambda_n}(s)\, ds \right\rangle = \int_{T_0}^{T} \langle z\mathbf{1}_{[T_0,t]}(s), \dot{u}_{\lambda_n}(s)\rangle\, ds,$$

we see that

$$\int_{T_0}^{t} \dot{u}_{\lambda_n}(s) ds \to \int_{T_0}^{t} h(s) ds \quad \text{weakly in } H.$$

As $(u_{\lambda_n}(t))_n$ converges strongly in H to $u(t)$, it results from the equality $u_{\lambda_n}(t) = a + \int_{T_0}^{t} \dot{u}_{\lambda_n}(s)\, ds$ that

$$u(t) = a + \int_{T_0}^{t} h(s) ds. \tag{3.12}$$

Consequently, $u(\cdot)$ is absolutely continuous with $\dot{u}(t) = h(t)$, for almost all t, and hence

$$\dot{u}_{\lambda_n}(\cdot) \to \dot{u}(\cdot), \text{ weakly in } L^2([T_0, T_0+\theta], H).$$

Set $z(t) := -f(t, u(t))$ and $I = [T_0, T_0+\theta]$. Keeping in mind that $z_\lambda(t) = -f(t, u_\lambda(t))$, we get

$$\dot{u}_{\lambda_n}(\cdot) - z_{\lambda_n}(\cdot) \to \dot{u}(\cdot) - z(\cdot), \text{ weakly in } L^2(I, H), \tag{3.13}$$

and by (3.9) we have $\|\dot{u}(t) - z(t)\| \leq \beta+\gamma$, and hence also $\|\dot{u}(t)\| \leq 2\beta+\gamma$, for almost every $t \in I$. The latter inequality entails by (3.12) that

$$\|u(t) - a\| \leq (t - T_0)(2\beta+\gamma) < \rho/3$$

since $\theta < \frac{\rho}{3(2\beta+\gamma)}$, so $u([T_0, T_0+\theta]) \subset B(a, \rho/3)$. Applying Mazur's lemma, through (3.13) there exist for each $n \in \mathbb{N}$ some integer $r(n) > n$ and real numbers $s_{k,n} \geq 0$ with $\sum\limits_{k=n}^{r(n)} s_{k,n} = 1,$ such that $\sum\limits_{k=n}^{r(n)} s_{k,n}(z_{\lambda_k} - \dot{u}_{\lambda_k})$ converges strongly to $z(\cdot) - \dot{u}(\cdot)$ in $L^2(I, H)$.

Extracting a subsequence, we may suppose that, for some negligible $N \subset I$, the derivatives $\dot{u}(t)$ and $\dot{u}_{\lambda_n}(t)$ exist for all $t \in I \backslash N$ and that

$$\sum_{k=n}^{r(n)} s_{k,n}(z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t)) \to z(t) - \dot{u}(t), \text{ for all } t \in I \setminus N. \tag{3.14}$$

We may also suppose that the inequalities in Lemma 3.4 hold for all $t \in I \backslash N$ and all λ_n with $n \in \mathbb{N}$. Fix any $t \in I \backslash N$. First we have by (3.8) and Schwartz inequality the estimation

$$\begin{aligned} &\left| \sum_{k=n}^{r(n)} s_{k,n} \langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t)) \rangle \right| \\ &\leq (\beta + \gamma) \sum_{k=n}^{r(n)} s_{k,n} ||u(t) - \operatorname{proj}_C(u_{\lambda_k}(t))||. \end{aligned} \tag{3.15}$$

Further, the Lipschitz continuity of $\operatorname{proj}_{C(t)}(\cdot)$ on $B(a, \frac{\varrho}{3})$ and the inclusion $u(t) \in C(t)$ (see (3.11)), ensure that

$$\operatorname{proj}_{C(t)}(u_{\lambda_n}(t)) - u(t) \underset{n \to +\infty}{\to} 0 \quad \text{strongly in } H,$$

so by (3.15)

$$\sum_{k=n}^{r(n)} s_{k,n} \langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_C(u_{\lambda_k}(t)) \rangle \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Writing for any $x' \in H$

$$\begin{aligned} &\sum_{k=n}^{r(n)} s_{k,n} \langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), x' - \operatorname{proj}_C(u_{\lambda_k}(t)) \rangle \\ &\quad = \left\langle \sum_{k=n}^{r(n)} s_{k,n}(z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t)), x' - u(t) \right\rangle \\ &\quad + \sum_{k=n}^{r(n)} s_{k,n} \langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), u(t) - \operatorname{proj}_C(u_{\lambda_k}(t)) \rangle \end{aligned}$$

we deduce from (3.14) and (3.16), as $n \to \infty$,

$$\sum_{k=n}^{r(n)} s_{k,n} \langle z_{\lambda_k}(t) - \dot{u}_{\lambda_k}(t), x' - \operatorname{proj}_C(u_{\lambda_k}(t)) \rangle \to \langle z(t) - \dot{u}(t), x' - u(t) \rangle. \tag{3.17}$$

On the other hand, noting from (2.2) and (3.8) that

$$z_\lambda(t) - \dot{u}_\lambda(t) \in N\big(C(t); \operatorname{proj}_{C(t)}(u_\lambda(t))\big),$$

we see by (2.1) that, for all $x' \in C(t)$,

$$\sum_{k=n}^{r(n)} s_{k,n}\langle z_{\lambda_k}(t) - \dot u_{\lambda_k}(t), x' - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t))\rangle$$
$$\leq \left(\frac{\beta+\gamma}{2\rho}\right)\sum_{k=n}^{r(n)} s_{k,n}||x' - \operatorname{proj}_{C(t)}(u_{\lambda_k}(t))||^2,$$

and since $\operatorname{proj}_{C(t)}(u_{\lambda_n}(t)) \to u(t)$ as $n \to \infty$, using (3.17) it follows that

$$\langle z(t) - \dot u(t), x' - u(t)\rangle \leq \left(\frac{\beta+\gamma}{2\rho}\right)||x' - u(t)||^2 \text{ for all } x' \in C(t),$$

which entails

$$-\dot u(t) + z(t) \in N^F(C(t); u(t)) = N(C(t); u(t)). \tag{3.18}$$

The inclusion being true for all $t \in I \setminus N$, the proof of the lemma is complete. □

The existence result over the whole interval $[T_0, T]$ can be obtained through the existence of the truncated interval above.

Lemma 3.7. *Assume that f is defined on $[T_0, T] \times H$ and satisfies for all $t \in [T_0, T]$ and $x, y \in H$*

$$\|f(t, x)\| \leq \beta \quad \textit{and} \quad \|f(t, x) - f(t, y)\| \leq k\|x - y\|.$$

Then a solution for the differential inclusion (E_f) can be obtained over the whole interval $[T_0, T]$ by subdivision of the latter into finitely many intervals of length less than $\frac{\rho}{3(2\beta+\gamma)}$.

Proof. Fix an integer $N \in \mathbb{N}$ such that $\frac{T-T_0}{N} < \frac{\rho}{3(2\beta+\gamma)}$. Without loss of generality we may then take for the positive real number θ (which has been fixed above with $\theta < \frac{\rho}{3(2\beta+\gamma)}$ the real number $\frac{T-T_0}{N}$, i.e., $\theta = (T - T_0)/N$. Put $T_i = T_0 + i\theta$ for $i = 0, 1, \cdots, N$. Lemma 3.6 provides a Lipschitzian solution u_1 (with $2\beta + \gamma$ as Lipschitz constant) on the interval $[T_0, T_1]$ for the differential inclusion

$$\begin{cases} -\dot u(t) \in N_{C(t)}(u(t)) + f(t, u(t)) \\ u(T_0) = a. \end{cases}$$

As $u_1(T_1) \in C(T_1)$ we may apply again Lemma 3.6 with T_1 in place of T_0 and with $u_1(T_1)$ as initial condition to obtain a Lipschitzian solution (with $2\beta+\gamma$ as Lipschitz constant) over the second closed interval $[T_1, T_2]$ for the differential inclusion

$$\begin{cases} -\dot u(t) \in N_{C(t)}(u(t)) + f(t, u(t)) \\ u(T_1) = u_1(T_1). \end{cases}$$

We can proceed in this way up to the last closed interval $[T_{k-1}, T_k]$. Defining the mapping $u(\cdot)$ on the whole interval $[T_0, T]$ by putting $u(t) := u_i(t)$ for any $t \in [T_{i-1}, T_i]$, with $i = 1, \cdots, N$, it is easily seen that $u(\cdot)$ provides a Lipschitzian (with $2\beta+\gamma$ as Lipschitz constant) solution over $[T_0, T]$ for the differential inclusion (E_f). □

The last step of the proof of Theorem 2.4 is the uniqueness of solution for (P) (in Lemma 3.6) and for (E_f). This feature is a consequence of the following lemma with U as $B(a, \rho/3)$ and H respectively.

Lemma 3.8. *Let U be an open set of H and let $T_1 \in]T_0, T]$. Assume that f is defined on $[T_0, T_1] \times U$ with values in H and satisfies the inequalities in (i) and (ii) of Theorem 2.4 for all $t \in [T_0, T_1]$ and $x, y \in U$. Then the differential inclusion*

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) + f(t, u(t)) \\ u(T_0) = a \in C(T_0) \end{cases}$$

has at most one solution $u(\cdot)$ over $[T_0, T_1]$ with $u([T_0, T_1]) \subset U$.

Proof. Let $u_i(\cdot) i = 1, 2$ be two solutions of the differential inclusion. By the ρ-hypomonotonicity property of the normal cone of $C(t)$ we have

$$\begin{aligned} &\langle \dot{u}_1(t) + f(t, u_1(t)) - \dot{u}_2(t) - f(t, u_2(t)), u_1(t) - u_2(t) \rangle \\ &\le \frac{1}{\rho} \big(\|\dot{u}_1(t) + f(t, u_1(t))\| + \|\dot{u}_2(t) + f(t, u_2(t))\| \big) \|u_1(t) - u_2(t)\|^2. \end{aligned}$$

Recalling that k is a Lipschitz constant of $f(t, \cdot)$ and putting

$$\alpha(t) := k + \frac{1}{\rho} \big(\|\dot{u}_1(t) + f(t, u_1(t))\| + \|\dot{u}_2(t) + f(t, u_2(t))\| \big),$$

we see through (ii) that, for almost every $t \in [T_0, T_1]$,

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \le \alpha(t) \|u_1(t) - u_2(t)\|^2,$$

or equivalently

$$\frac{d}{dt} \|u_1(t) - u_2(t)\|^2 \le 2\alpha(t) \|u_1(t) - u_2(t)\|^2.$$

Further, by the assumption (ii) the non-negative function $\alpha(\cdot)$ is Lebesgue integrable on $[T_0, T_1]$. Then, the Gronwall lemma (see Lemma 3.2) guarantees that, for all $t \in [T_0, T_1]$,

$$\|u_1(t) - u_2(t)\|^2 \le \|u_1(T_0) - u_2(T_0)\|^2 \exp \big(2 \int_{T_0}^{t} \alpha(\tau) \, d\tau \big) = 0,$$

which justifies the uniqueness property. □

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