For a real function $f : \mathbb{R} \to \mathbb{R}$ we denote by $f'_+(t)$ the right derivative of $f$ at $t$ when it exists, that is,

$$f'_+(t) = \lim_{r \searrow 0} \frac{f(t + r) - f(t)}{r}.$$ 

**Lemma 0.1** Let $h, v \in X$, with $\|v\| > 1$ and $\|h\| = 1$ and $v$ and $h$ are linearly independent, $\beta > 1$ and $s \geq 0$. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \|tv + \beta h\| - \|tv + 2sv + h\|, \quad g(t) = \|tv - \beta h\| - \|tv + 2sv - h\|.$$ 

Then $f$ and $g$ are differentiable and satisfying:

- $f$ is non-decreasing on $] - \infty, -\frac{2s\beta}{\beta - 1}]$ ($f'_+(t) \geq 0$, $\forall t \leq 0$) and non-increasing on $[-\frac{2s\beta}{\beta - 1}, +\infty[$ ($f'_+(t) \leq 0$, $\forall t \geq -\frac{2s\beta}{\beta - 1}$).
- $g$ is non-decreasing on $] - \infty, -\frac{2s\beta}{\beta - 1}]$ ($g'_+(t) \geq 0$, $\forall t \leq -\frac{2s\beta}{\beta - 1}$) and non-increasing on $[-\frac{2s\beta}{\beta - 1}, +\infty[$ ($g'_+(t) \leq 0$, $\forall t \geq -\frac{2s\beta}{\beta - 1}$).
- For all $t \geq 0$

$$f(t) \geq (\beta - 1)v'(v; h) - 2s\|v\|, \quad \text{and} \quad g(t) \geq (\beta - 1)v'(v; -h) - 2s\|v\|,$$

where $v'(v; \cdot)$ denotes the usual directional derivative of the convex function $\nu := \|\cdot\|$.

**Proof.** We study only the function $f$ since the result for $g$ is obtained in changing $h$ in $-h$. It is not difficult to see that $v'(rv; v) = v'(u; v)$ for all $u, v \in X$. So, for all $t \neq 0$,

$$f'_+(t) = v'(tv + \beta h; v) - v'(tv + 2sv + h; v) = v'\left(\frac{t}{\beta}v + h; v\right) - v'(tv + 2sv + h; v).$$

From the continuity of the convex function $\nu$, choose $x^*_1 \in \partial \nu\left(\frac{t}{\beta}v + h\right)$ and $x^*_2 \in \partial \nu(tv + 2sv + h)$ such that

$$\langle x^*_1, v \rangle = v'\left(\frac{t}{\beta}v + h; v\right) \quad \text{and} \quad \langle x^*_2, v \rangle = v'(tv + 2sv + h; v).$$

Consequently,

$$f'_+(t) = \frac{\beta}{(1 - \beta)t - 2s\beta} \langle x^*_1 - x^*_2, \frac{t}{\beta}v + h \rangle - \langle tv + 2sv + h, v \rangle,$$

hence the monotonicity of the subdifferential $\partial \nu$ ensures that

$$f'_+(t) \geq 0 \quad \text{for all} \; t \leq -\frac{2s\beta}{\beta - 1} \quad \text{and} \quad f'_+(t) \leq 0 \quad \text{for all} \; t \geq -\frac{2s\beta}{\beta - 1}.$$ 

We deduce that $f$ is non-decreasing on $] - \infty, -\frac{2s\beta}{\beta - 1}]$ and non-increasing on $[-\frac{2s\beta}{\beta - 1}, +\infty[$. The last item results from the following

$$\lim_{t \to +\infty} f(t) \geq -2s\|v\| + \lim_{\tau \to 0^+} \frac{\|v + \tau \beta h\| - \|v\| - \|v + \tau h\| - \|v\|}{\tau} = -2s\|v\| + \beta v'(v; h) - v'(v; h) = (\beta - 1)v'(v; h) - 2s\|v\|.$$

The following lemma whose proof is similar to that of Lemma 2.2 in [1] will be used later.

**Lemma 0.2** Let $\varepsilon > 0$, $u, h \in X$, with $\|u\| = 1$ and $0 < \|h\| \leq \min(1, \frac{\varepsilon}{2})$, satisfying

$$\|u + h\| > \|u - h\|, \quad \frac{\|u + h\| + \|u - h\| - 2}{\|h\|} \geq \varepsilon. \quad (0.1)$$

Then there exists $s \in [0, \|h\|]$ such that

- $\|u + h - su\| = \|u - h + su\|$ and
- $\frac{\|u + h - su\| - 1}{\|h - su\|} \geq \frac{\varepsilon}{8}$.  

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Theorem 0.1 Let $X$ be a Banach space. Then the following assertions are equivalent:

1. The norm $\| \cdot \|$ of $X^*$ is W* UR.

2. For each closed set $S \subset X$, and all $a, b \in S$, with $a \neq b$, we have
   \[
   \lim_{\|x\| \to +\infty} \inf (d_{[a,b]}(x) - d_S(x)) \geq 0.
   \] (0.2)

3. For all $a, b \in X$, with $a \neq b$, we have
   \[
   \lim_{\|x\| \to +\infty} \inf (d_{[a,b]}(x) - \min\{\|x - a\|, \|x - b\|\}) \geq 0.
   \] (0.3)

4. The norm $\| \cdot \|$ is UGD.

Comment. We know that $1. \iff 4.$ (see the book by Deville, Godefroy and Zisler) and $4. \implies 2.$ (this is exactly our Proposition 4.2). Here we give another proof of the latter implication.

Proof. 1. $\implies$ 2.: Suppose that the assertion 2. is not satisfied. Then there exist $\mu > 0$, a closed set $S \subset X$, and $a, b \in S$, with $a \neq b$, and a sequence $(x_n) \subset X$, with $\|x_n\| \to +\infty$ such that for $n$ large enough
   \[
   \lim_{\|x\| \to +\infty} (d_{[a,b]}(x_n) - d_S(x_n)) < -\mu.
   \] (0.4)

Fix $n$ sufficiently large satisfying (0.4). Let $p_n \in [a, b]$ such that $d_{[a,b]}(x_n) = \|x_n - p_n\|$. Fix $x^*_n \in - \partial d_{[a,b]}(p_n)$, so $\|x^*_n\| = 1$ for $n$ large enough since $x_n \notin [a, b]$ according to $\|x_n\| \to +\infty$. Observe that, for all $p \in [a, b]$,
   \[
   \langle -x^*_n, p - x_n \rangle \leq d_{[a,b]}(p) - d_{[a,b]}(x_n) = -d_{[a,b]}(x_n).
   \] (0.5)

For all $p \in [a, b]$, it ensues that
   \[
   \langle -x^*_n, p - p_n \rangle = \langle -x^*_n, p - x_n \rangle + \langle -x^*_n, x_n - p_n \rangle - d_{[a,b]}(x_n) + \|x_n - p_n\| = -d_{[a,b]}(x_n) + d[a,b](x_n) = 0.
   \]

This and the inclusion $p_n \in [a, b]$ for $n$ large enough (due (0.4)) $p_n \in [a, b]$ entail
   \[
   (x^*_n, a - b) = 0.
   \] (0.6)

Further, from (0.5) we also have
   \[
   \langle -x^*_n, p_n - x_n \rangle \leq -d_{[a,b]}(x_n) = -\|x_n - p_n\|
   \]

hence
   \[
   \|x_n - p_n\| \leq \langle x^*_n, p_n - x_n \rangle.
   \] (0.7)

On the other hand, since $\|p_n - x_n\| - \|a - x_n\| \leq -\mu$, then, for all $a^*_n \in \partial \| \cdot - x_n \| (a)$, we have
   \[
   \mu \leq \|a - x_n\| - \|p_n - x_n\| \leq \langle a^*_n, a - p_n \rangle.
   \] (0.8)

Let $t_n \in [0, 1]$ be such that $p_n = t_n a + (1 - t_n) b$. Using relation (0.4), we find $s_1, s_2 \in [0, 1]$ such that for all $n$ sufficiently large $s_1 < t_n < s_2$. As $a - p_n = (1 - t_n)(a - b)$, relation (0.8) ensures that
   \[
   \frac{\mu}{1 - s_1} \leq \frac{\mu}{1 - t_n} \leq \langle a^*_n, a - b \rangle
   \]

which implies (via (0.6))
   \[
   \frac{\mu}{1 - s_1} \leq \langle a^*_n - x^*_n, a - b \rangle.
   \] (0.9)

On the other hand, relations (0.7) and (0.6) ensure that
   \[
   \|p_n - x_n\| + \|a - x_n\| \leq \langle x^*_n, p_n - x_n \rangle + \langle a^*_n, a - x_n \rangle
   \]

   \[
   = \langle x^*_n, t_n(a - b) + b - x_n \rangle + \langle a^*_n, a - x_n \rangle
   \]

   \[
   = \langle x^*_n, b - x_n \rangle + \langle a^*_n, a - x_n \rangle
   \]

   \[
   = \langle x^*_n, a - x_n \rangle + \langle a^*_n, a - x_n \rangle
   \]

   \[
   = \langle x^*_n + a^*_n, a - x_n \rangle
   \]

   \[
   \leq \|x^*_n + a^*_n\| \cdot \|a - x_n\|.
   \]
Hence
\[ \frac{\|p_n - x_n\| + \|a - x_n\|}{\|a - x_n\|} \leq \|x_n^* + a^*_n\|. \]

Thus
\[ \|x_n^*\| = \|a^*_n\| = 1, \quad \lim_{n \to +\infty} \|x_n^* + a^*_n\| = 2 \]

and by (0.9) we have
\[ \liminf_{n \to +\infty} (a^*_n - x_n^*, a - b) \geq \frac{\mu}{1 - s_1}. \] (0.11)

It results that relations (0.10) and (0.11) contradict the W*UR property of \( X^* \).

2. \( \implies \) 3. : It is obvious.

3. \( \implies \) 4. : NOTE THAT relation 3. ensures the Gâteaux differentiability of the norm. OK

Suppose that the norm \( \| \cdot \| \) is not UGD. Then there exist \( b \in X \), with \( \|b\| = 1, \varepsilon > 0 \) and sequences \((u_n) \subset X\) with \( \|u_n\| = 1 \) for all \( n \) and \((t_n) \subset ]0, +\infty[ \) with \( \lim_{n \to +\infty} t_n = 0 \) such that, for all \( n \in \mathbb{N} \),
\[ \frac{\|u_n + t_n h\| + \|u - t_n h\| - 2}{t_n} \geq \varepsilon \] (0.12)

The proof of this implication is down in two steps.

**Step1.** For infinitely many integers \( n \), the equality
\[ \|u_n + t_n h\| = \|u_n - t_n h\| \]
holds true. Put \( a = h \) and \( b = -h \) and \( x_n = \frac{u_n}{t_n} \). Then
\[ \lim_{n \to +\infty} \|x_n\| = +\infty, \quad d_{[a,b]}(x_n) \leq \|x_n\| \]
and
\[ \min\{\|x_n - a\|, \|x_n - b\|\} = \|x_n - a\|. \]

Relation (0.12) ensures that
\[ d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} \leq \|x_n\| - \|x_n - a\| = \frac{1}{t_n} - \frac{1}{t_n} \|u_n - t_n h\| \]
\[ = \frac{2}{t_n} \left( 2 - \frac{\|u_n - t_n h\| - \|u_n + t_n h\|}{t_n} \right) \leq - \frac{\varepsilon}{2} \]
and this contradicts the assumption 3.

**Step2.** The equality holds true only for a finite number of integers \( n \). Without loss of generality, suppose that
\[ \forall n \in \mathbb{N}, \quad \|u_n + t_n h\| > \|u_n - t_n h\|. \]
Applying Lemma 0.2, with \( h = t_n h \) and \( u = u_n \), there exists \( s_n \in ]0, t_n[ \) such that
\[ \|u_n + t_n h - s_n u_n\| = \|u_n - t_n h + s_n u_n\| \] (0.13)
\[ \|u_n + t_n h - s_n u_n\| = \frac{\|u_n + t_n h - s_n u_n\| - 1}{\|t_n h - s_n u_n\|} \geq \frac{\varepsilon}{8}. \] (0.14)

Relations (0.13) and (0.14) assert that \( u_n \) and \( b \) are linearly dependent for at most a finite number of integers \( n \) (THIS NEEDS TO BE ARGUED). So we may assume that for all \( n \in \mathbb{N} \), \( u_n \) and \( h \) are linearly independent.

Put \( v_n = \frac{u_n}{t_n} \) and \( x_n = (1 - s_n) v_n \). Two cases may occur:

**Case I :** \( \lim_{n \to +\infty} \|b - s_n v_n\| = 0. \) Without loss of generality we may assume that \( \lim_{n \to +\infty} \|h - s_n v_n\| = 0. \) In this case, since \( \|h\| = \|u_n\| = 1 \),
\[ \lim_{n \to +\infty} \frac{s_n}{t_n} = 1. \] (0.15)
Put \( a = \frac{3}{2}h \) and \( b = -\frac{3}{2}h \). Apply Lemma 0.1, with \( \beta := \frac{3}{2} \), \( t := 1 - s_n \) and \( \nu = v_n \), the last item of this lemma gives, with \( s := 0 \) in the first inequality of that item and \( s := s_n \) in the second inequality,

\[
\|(1 - s_n)v_n + \frac{3}{2}h\| \geq \|(1 - s_n)v_n + h\| + \frac{1}{2}\nu'(v_n; h)
\]

and

\[
\|(1 - s_n)v_n - \frac{3}{2}h\| \geq \|(1 + s_n)v_n - h\| + \frac{1}{2}\nu'(v_n; h) - 2s_n\|v_n\|.
\]

Taking into account relation (0.13), we obtain

\[
\|x_n - a\| \geq \|(1 + s_n)v_n - h\| + \frac{1}{2}\nu'(v_n; h) - 2s_n\|v_n\| \geq \|(1 + s_n)v_n - h\| - \frac{1}{2} - 2s_n\|v_n\|
\]

and

\[
\|x_n - b\| \geq \|(1 + s_n)v_n - h\| + \frac{1}{2}\nu'(v_n; h) \geq \|(1 + s_n)v_n - h\| - \frac{1}{2}.
\]

Consequently

\[
\min\{\|x_n - a\|, \|x_n - b\|\} \geq \|(1 + s_n)v_n - h\| - \frac{1}{2} - 2s_n\|v_n\|.
\]

Since \( d_{[a,b]}(x_n) \leq \|x_n\| = \frac{1 - s_n}{t_n} \), relations (0.14) and (0.16) ensure that

\[
d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} \leq \frac{1 - s_n}{t_n} - \|(1 + s_n)v_n - h\| + \frac{1}{2} + 2s_n\|v_n\|
\]

\[
\leq -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| - \frac{s_n}{t_n} + \frac{1}{2} + 2s_n\|v_n\|
\]

\[
= -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| + \frac{s_n}{t_n} + \frac{1}{2}.
\]

**FROM THE LATTER NO DIRECT CONTRADICTION IS OBTAINED**

**Case II:** \( \liminf_{n \to +\infty} \| h - s_n v_n \| = \alpha > 0 \). Without loss of generality we may assume that

\[
\lim_{n \to +\infty} \| h - s_n v_n \| = \alpha.
\]

Put \( \beta = 1 + \frac{s_n}{t_n} \), \( a = \beta h \) and \( b = -\beta h \). Apply once again Lemma 0.1, with \( s = s_n \) and \( \nu = v_n \), and taking into account relation (0.13) we get from the last item of this lemma with \( t := 1 - s_n \) and \( s := s_n \),

\[
\|x_n - a\| \geq \|(1 + s_n)v_n - h\| + (1 - \beta)\nu'(v_n; h) - 2s_n\|v_n\|
\]

and

\[
\|x_n - b\| \geq \|(1 + s_n)v_n - h\| + (\beta - 1)\nu'(v_n; h).
\]

It ensues that

\[
\min\{\|x_n - a\|, \|x_n - b\|\} \geq \|(1 + s_n)v_n - h\|(1 - \beta) - 2s_n\|v_n\|.
\]

Since \( d_{[a,b]}(x_n) \leq \|x_n\| = \frac{1 - s_n}{t_n} \), relations (0.14) and (0.18) ensure that

\[
d_{[a,b]}(x_n) - \min\{\|x_n - a\|, \|x_n - b\|\} \leq \frac{1 - s_n}{t_n} - \|(1 + s_n)v_n - h\| + (\beta - 1) + 2s_n\|v_n\|
\]

\[
= -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| - \frac{s_n}{t_n} + (\beta - 1) + 2s_n\|v_n\|
\]

\[
\leq -\frac{\varepsilon}{8}\|h - \frac{s_n}{t_n}u_n\| + \frac{s_n}{t_n} + \frac{\varepsilon\alpha}{16}.
\]

**HERE ALSO NO CONTRADICTION IS DIRECTLY OBTAINED**

**References**